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The use of Van Hiele's geometric thinking model to interpret Grade 12 learners' learning difficulties in Euclidean Geometry

Abstract

The 21st-century mathematics classrooms should equip learners with well-grounded knowledge and thinking skills pertaining to geometry. However, Euclidean geometry remains one of the challenging, if not the most difficult topic for many learners. As a result, the purpose of this article is to interpret Grade 12 learners' learning difficulties in Euclidean geometry. We use Van Hiele's geometric thinking model and Hoffer's skills to argue an interpretation of learning difficulties in Euclidean geometry as a focal point towards creating effective teaching and learning of this important topic. This explanatory sequential mixed-methods approach involved 60 Grade 12 learners who wrote a geometry test and completed a questionnaire based on Van Hiele's geometric thinking levels. In addition, semi-structured interviews were conducted with a sample of 12 learners and four educators to investigate their views about geometry learning difficulties. The findings of the study revealed that learners had poor conceptualisation of properties of shapes, visualisation skills, circle theorems and geometry terminology, resulting in them experiencing learning difficulties. The recommendations are that, during instruction learners should be given the opportunity to manipulate real geometric objects to enhance their visualisation and visual thinking skills. In addition, we recommend that educators should teach level-specific geometry vocabulary to enable learners to understand concepts at different Van Hiele's levels. Furthermore, we recommend that educators should use constructivist teaching approaches that encourage learners' conceptual understanding instead of traditional methods that promote rote memorisation of geometric facts. Educators should develop learners' broad knowledge of geometry to overcome geometry-related errors and misconceptions.

Keywords: *Euclidean geometry, geometry content knowledge, informal deductive reasoning, learning difficulties, visualisation*

1. Introduction and background

The 21st-century mathematics classrooms should equip learners with well-grounded knowledge and thinking skills pertaining to geometry. Euclidean geometry, which consists of theorems related to the circle and their application in problem-solving (DBE, 2011; Kgopane, 2021; Mabotja,

2017), is essential in engineering science-related careers such as civil engineering and architecture. Furthermore, Euclidean geometry is considered an essential field of study that equips learners with spatial visualisation (Alex & Mammen, 2016; Jones & Tzekaki, 2016; Bayuningsih, Usodo & Subanti, 2018). In addition, learners develop geometric reasoning, justification, and proof skills through learning Euclidean geometry (Chuene *et al*., 2023; Dhlamini *et al*., 2019; Mabotja *et al*., 2018). These skills are often considered as prerequisites in science; technology, engineering and mathematics (STEM)-related careers such as engineering and architecture (Blacklock, 2018; Ubah & Bansilal, 2019). Hence, one of the specific skills as outlined in the Curriculum Assessment Policy Statement (CAPS) mathematics curriculum is the development of learners' abilities to use spatial skills, properties of shapes and objects to identify, pose and solve problems creatively and critically (DBE, 2011). By so doing, learners will be able to generalise, make conjectures, and try to justify their reasoning in a variety of problem-solving contexts and hence demonstrate learning with understanding (Kgopane, 2021; Chuene *et al*., 2023).

DBE (2011) regards geometry as an important topic in the Further Education and Training (FET) mathematics curriculum. This assertion is evident in that Euclidean geometry accounts for over 30% weighting of Paper 2 in FET mathematics assessments (DBE, 2011). Herein, learners could pass Paper 2 by passing Euclidean Geometry alone due to the pass mark for Paper 2 being 30%. However, over the past years, research has shown that the teaching and learning of Euclidean geometry are characterised by various challenges (Chuene et al., 2023; Maqoqa, 2024; Ubah & Bansilal, 2019). For instance, previous National Senior Certificate (NSC) examination diagnostic reports show that learners' performance in the Euclidean geometry is exceedingly poor when compared to other topics in the Grade 12 mathematics examinations (DBE, 2021). In this vein, previous studies mainly attribute learners' dismal performance in geometry to educators' inadequate knowledge of geometry (Machisi, 2021; Novak & Tassell, 2017; Robertson & Graven, 2019; Tachie, 2020; Zamisa, 2019).

Furthermore, most educators' dependency on transmission-mode teaching strategies is often viewed as a contributing factor to learners' learning difficulties in Euclidean geometry (Bosman & Schulze, 2018; Masilo, 2018; Tachie, 2020). Such teaching strategies are characterised by telling learners the properties of shapes and expecting them to use the facts to complete exercises without understanding the underlying concepts. The contention is that transmission-mode teaching strategies do not promote learners' active participation and growth of geometry understanding and reasoning (Chuene *et al*., 2023; Mabotja *et al*., 2018).

Previous studies conducted by Masilo (2018) and Yalley *et al*. (2021) found that the use of Van Hiele's geometry theory for instructional decisions has a positive effect on learners' performance in geometry. An earlier study by Alex and Mammen (2018) argues that many Grade 12 learners in South Africa remain fixed at Van Hiele's concrete visualisation level instead of the expected abstract reasoning level in geometry. Similarly, Yalley *et al*. (2021) found that learners who were at the precognition level before the intervention improved to Van Hiele's level 2 after the intervention. Herein, the contention is that a Van Hiele-based teaching approach facilitates learners' effective understanding of Euclidean geometry concepts.

While Van Hiele's model has been used in quasi-experimentally research designs to measure its effectiveness on performance (Yalley *et al*., 2021), we use this model to interpret learners' learning difficulties at various geometric thinking levels. Thus, this study did not seek to use a particular approach or teaching strategy to improve learners' performance, but

focused on the interpretation of specific learning difficulties encountered by Grade 12 learners. In this article, we argue that an interpretation of learning difficulties in Euclidean geometry is a focal point towards creating effective learning of this important topic. When we attach meaning to difficulties experienced by Grade 12 learners, we can find ways of tailoring our instructional decisions to circumvent such difficulties. Thus, the interpretation of real learning difficulties has a direct bearing on learners' geometry proficiency (Alex & Mammen, 2018; Mudhefi, 2022) and provides opportunities for identifying learners' misconceptions and errors (Mackle 2017; Mosia *et al.,* 2023). As a result, we use Van Hiele's (1986) geometric thinking model and Hoffer's (1981) five basic skills to interpret the Euclidean geometry learning difficulties prevalent amongst Grade 12 learners.

2. Theoretical framework

As previously mentioned, this study uses Van Hiele's (1986) geometric thinking model to foreground learning difficulties experienced by Grade 12 learners. Van Hiele's model explains the development of geometrical thinking through hierarchal five levels (Van Hiele, 1986). These levels are shown in Figure 1 below.
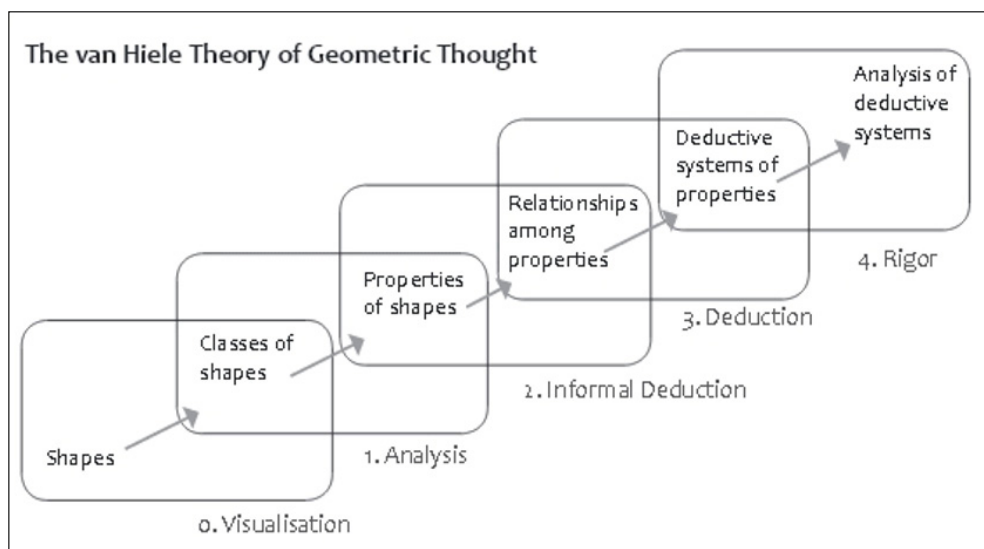

Figure 1: Van Hiele's theory of geometric thought (Van de Walle, 2004: 347)

The hierarchical nature of the model seeks to illustrate that learners' knowledge of geometry in the first layers serves as a foundation to succeed at the next level (Mudhefi, 2022; Van de Walle, 2016; Yalley *et al.*, 2021). The contention is that students' ability to build a concrete understanding of geometry is sequential. This suggests that students cannot develop mastery of the next level if their knowledge of the prior level is not developed as earlier argued by Jones (2002) and Van Hiele (1999). Nevertheless, the hierarchy does not necessarily imply that learners who have strong visualisation (level 0) abilities will automatically not experience difficulties at the next level.

In this study, we used Van Hiele's (1986) geometric thinking model as a lens to map learners' thinking levels. However, it is in the theoretical work of Hoffer's (1981) geometric thinking skills that we interpreted learning difficulties encountered by Grade 12 learners in

Euclidean geometry. Hoffer's (1981) model identified the following skills; firstly, *visual 'skill of the eye'* – learners' ability to recognise and interpret diagrams geometrically (Mudhefi, 2022). Secondly, *verbal skills* are characterised by learners' ability to read, comprehend and communicate geometric information verbally (Hoffer, 1981), while thirdly, *drawing skills* help learners to understand relationships between shapes and their properties through hands-on activities (Mudhefi, 2022; Patkin & Sarfaty, 2012), Fourthly, *logical skills* encompass learners' abilities to identify similarities and differences between shapes and their resultant ability to logically link geometric statements to correct reasons. Lastly, *application skills* emphasise learners' ability to apply geometric thinking in different geometric problem-solving situations (Hoffer, 1981; Mudhefi, 2022).

Hoffer's model guided our interpretation in that it enabled us to link the identified learning difficulties to a set of deficient skills in terms of the learners' conceptualisation of geometry at each Van Hiele level. Hoffer's skills guided us in suggesting possible causes for the identified learning difficulties. For example, difficulties at visualisation level were interpreted in terms of learners' poorly developed observational and mental imaging skills. Similarly, challenges encountered at analysis and informal deductive levels are concerned with the learners' inability to comprehend and verbalise geometric content and apply logical reasoning skills, respectively.

3. Methodology

This article uses parts of the first author's master's study by adopting an explanatory sequential mixed-methods approach. Creswell and Poth (2016) view it as a sequential combination of quantitative and qualitative research approaches in a single study. In this vein, the explanatory mixed-methods approach was used to ensure that data evaluation captures statistical information, which was interpreted using participants' views and opinions (Shannon-Baker, 2016). The sequential implementation of quantitative and qualitative methods enabled the researchers to explain and contextualise the findings. As a result, the explanatory sequential mixed-methods approach was deemed suitable for this study because of its effectiveness in using qualitative data to explain and contextualise existing quantitative findings. In this article, quantitative data were interpreted qualitatively through identifying, describing, analysing and presenting naturalistic summaries of learners' learning difficulties in geometry.

The participants in the study were 60 Grade 12 learners randomly selected from six purposively sampled schools in the Ngaka Modiri Molema District, North-West Province. Herein, we randomly sampled the study participants from purposefully selected schools to ensure learners from targeted schools had an equal chance of being selected to participate in the study. The 60 all learners wrote a Van Hiele level-based Euclidean geometry test and completed a questionnaire based on items about geometry as part of the data-collection methods. In addition, face-to-face interviews were conducted with 12 learners, three learners from four of the sampled schools to ascertain learners' perspective on learning difficulties in geometry. In addition, four educators sampled from the target schools also participated in interview sessions to express their views on challenges encountered by learners in the teaching and learning of geometry.

On the other hand, ethics as a moral and professional imperative ensures that confidentiality is at the core of any research (McMillan & Schumacher, 2014). In view of this study, ethical requirements were satisfied by signing a code of ethical conduct that binds researchers to strict protocols, methods and guidelines for conducting education research.

Thus, researchers signed a code of ethical conduct pledging to abide by all ethical protocol at all stages of the research process. Similarly, confidentiality of participants' information was achieved by ensuring that their names and identities would not be revealed at any stage of the research, from data collection to the reporting of the results. Furthermore, participants were assured that their participation is voluntary, and they had the freedom to terminate it any time they felt they could not proceed without giving reasons for the termination. Parents whose children were minors were asked to sign consent forms as confirmation that they allowed their children to take part in the research.

To ensure the quality of research instruments, the first author sought the assistance of experienced mentors (second and third authors), who helped by balancing the instruments' validity and reliability to data-collection procedures, interventions, measured variables and the underlying theoretical constructs and elements (Creswell, 2014). Reliability and validity were ensured by piloting the instruments with a few learners and then adjustments were made with the help of experienced mentors in terms of the questions' structure and time allocation to suit learners of different abilities.

Furthermore, the researchers ensured that quality criteria were achieved through consulting mathematics heads of department and subject specialists at the Department of Education for the Ngaka Modiri Melema District. These experts assisted by ensuring that there is a balance between participants, the research instruments and the procedure for collecting data (construct validity). The researchers ensured the trustworthiness of the data-collection process by triangulating qualitative and quantitative data (Creswell, 2014) and conducting members-checking credibility (Merriam & Tisdell, 2016).

4. Results and discussions

Firstly, data analysis of the written test involved categorising learners by numbers and percentages into those who passed and those who failed at 50% on the test questions set at the first three Van Hiele levels. On the other hand, questionnaire data were used to complement the test results as learners gave justification for the geometric challenges they encountered in the written test. Below is a table that was used to present the test results at visualisation level.

Table 1: Frequency distribution matrix to test question 1, at Van Hiele's visualisation level (0)

Van Hiele Level	Learners who passed at basic level	(%) passed	Learners who failed at basic level	(%) failed
Visualisation level (0)	37	62%	23	38%

The data as presented above indicate that 62% of the learners passed and 38% failed at Van Hiele's visualisation level. The percentages above were arrived at by comparing the number of learners who scored below 50% in question 1 of the test to those who scored above 50% at visualisation level. In support of this data, learners' questionnaire responses revealed that those who did not pass at visualisation level could not use their intuition to recognise and name angles between parallel lines.

Table 2:   A frequency distribution matrix of the learners' questionnaire responses to pre-set questions at Van Hiele's level (0)

Visualisation level 0		Learners who Strongly Agreed	Learners who Agreed	Learners who Disagreed	Learners who Strongly Disagreed
2.1	I have challenges with naming different lines, angles, triangles and quadrilaterals	4 (6%)	16 (28%)	31 (57%)	9 (15%)

An overview of the data in Table 2 suggests that 20 learners (34%) either agreed or strongly agreed that they had difficulties with recognising and naming geometric shapes, whereas a total of 40 (66%) or strongly disagreed to having difficulties with basic geometrical knowledge. Therefore, we concluded that some learners still have challenges in acquiring and applying basic visualisation skills.

As a result, data above were supported by HoDs' questionnaire responses as indicated in the frequency distribution below:

Table 3:   Frequency distribution matrix of HoDs' questionnaire responses to question 1 at Van Hiele's visualisation level 0

Visualisation level 0 sample questions		Strongly Agree	Agree	Disagree	Strongly Disagree
1.3	Learners fail to use intuition to recognise and name geometrical figures.	1	2	1	0
2.1	Development of learners' spatial skills is overlooked by educators.	1	3	0	0
3.5	Learners fail to link their visual to verbal skills to recognise geometrical diagrams.	0	3	1	0

Data gathered indicate that about 70% of their interview responses confirmed the idea that learners have challenges with understanding the spatial orientation of shapes because they lack flexibility and encode geometric information in terms of fixed attributes. Similarly, learners' questionnaire responses showed that (68%) of them experienced challenges related to the accommodation and assimilation of visual information. Thus, the data gathered suggest that learners struggle to acquire visual recognition skills. That was confirmed by about 75% of the HoDs' questionnaire responses, which suggested that most learners who failed questions set at basic visualisation were more likely going to experience difficulties in transitioning from visual to verbal representation.

Data were analysed to help in answering the research question "What are the Grade 12 learners' learning difficulties related to analysis level in Euclidean geometry?" In this vein, learners were required to identify and describe circle components from the diagram using formal geometry language. Thus, the main focus at this level was to identify learners' challenges related to way they interpret geometrical figures using formal geometry language. The results were presented as indicated below.

Table 4:   A frequency distribution matrix of learner performance to test questions set at Van Hiele's analysis level 1

Van Hiele's level	Learners who passed at analysis level	(%) passed	Learners who failed at analysis level	(%) failed
Analysis level 1	27	45%	33	55%

The learners' results, as presented above, revealed that 55% of them failed test questions set at Van Hiele's analysis level. The same results indicated that only 45% showed good mastery of descriptive skills at this level. The data gathered by the researchers suggest that most learners have difficulties in using formal geometry language to describe the components of a circle, hence compromising their ability to conceptualise theorems at the next Van Hiele level. This data were further supported by educators' responses to the questionnaire at analysis level as indicated in the frequency distribution matrix below:

Table 5:   A frequency distribution matrix of educators' questionnaire on test questions set at Van Hiele's analysis level

Analysis level 1 sample questions		Strongly Agreed	Agreed	Disagreed	Strongly Disagreed
2.1	Grade 12 learners' level of comprehension is lower than the geometry terminology level.	(66%)	(26%)	(8%)	(8%)
2.2	Learners cannot interpret geometry language correctly.	(51%)	(25%)	(17)	(8%)
2.4	Learners have difficulties linking geometric shape properties to visual representations	(68%)	(15%)	(8%)	(8%)
	Average responses (%)	(60%)	(22%)	(10%)	(8%)

An overview of the average responses showed that 82% of the educators were of the view that most learners have a language barrier in explaining the properties of shapes, with only 18% disagreeing with that notion. Therefore, we concluded that most learners have difficulties in integrating the visualisation skills acquired at Van Hiele level 0 with the descriptive skills at analysis level 1. We therefore concluded that most learners were likely to have challenges with informal deductive reasoning at the next level of Van Hiele's hierarchy of geometric thinking. This view is based the on cumulative nature of understanding geometry as suggested by Van Hiele's model.

Similarly, learners' interview responses confirmed that most of them have difficulties with geometry terminology, properties of shapes, writing correct reasons to geometric statements. For example, L22 acknowledged his

> *challenge is giving reasons because I write reasons that are not used in other sides and angles. Sometimes … reasons which are not supposed to be given in geometric shapes.*

Similarly, another learner attributed her confusion and difficulties on complex geometry diagrams as evident in the following comment:

> *test had a lot of complicated diagrams, and the theorems were too confusing. I may know the relationships between properties, but sometimes I forget how to write the reasons or theorem correctly* (L39).

For learner L25,

> teachers give us less examples and they just stick to what they want to teach and that makes it hard for us to understand. I also have challenges in arranging statements, but I sometimes get correct reasons.

L25 further explained,

> The teachers are too fast and do not have time to demonstrate and that is why we get wrong. We need a lot of time during the lessons.

Learners' responses, as revealed in these extracts, suggest the existence of learning difficulties related to their conceptualisation of relationships among properties of shapes. In addition, these responses suggest that learners lack basic understanding theorems, hence they resort to rote memorisation of geometric facts. Similarly, interview responses indicate that learners experience difficulties with writing correct reasons to geometric statements. Some learners blamed their lack of understanding on poor teaching methods.

Furthermore, both quantitative and qualitative data were analysed to the research question:

What are the Grade 12 learners' learning difficulties related to informal deduction level in Euclidean Geometry?

In answering this question, learners were deemed to have learning difficulties if they failed to:

- Apply theorems about property relationships of a variety of geometrical shapes.

- Calculate missing angles and make short deductions based on logical thinking.

- Use correct reasons to justify geometric statements

In this regard, the test results were analysed as follows:

Table 6: A frequency distribution matrix of learner performance at Van Hiele's informal deductive level 2

Learner performance distribution at informal deductive level	Learners who passed at basic level	(%) passed	Learners who did not pass at basic level	(%) not passed
Informal deductive thinking	13	22%	47	78%

The learner performance data as presented in the table indicate that only 22% passed the test at 50% at informal deductive level and 78% of the learners did not achieve the 50% target level. Test results showed that most of the learners who failed at this level could not relate properties of shapes within diagrams. These figures suggest that learners do experience difficulties in determining appropriate property relationships, as more lines are included in a geometric diagram. Learners' written test responses showed that most learners struggled to calculate the missing angles from the diagrams, suggesting their inability to relate properties of geometric shapes to relevant theorems.

These test results were confirmed by both learners and educators' interview responses to learning difficulties at Van Hiele's informal deductive level as shown below:

> *Sometimes the diagrams are complicated and identifying the theorems to use is challenging. I can only memorise the theorems but don't know when and how to use them. The teachers should try to use different teaching methods such as applying practical lessons* (L37).

> *Learners do not seem to have enough exposure to geometry riders. They need to practice diagram analysis in lower grades. It seems learners with poor English background find it difficult to grasp geometry language. It could also be......not exposed to the language in lower grades. I think learners are not exposed enough to solving riders. I will recommend geometry workbooks for learners from grade 10 to 12* (E4).

> *Learners lack knowledge of relating concepts in a logical way which lead to proof. Learners start with what they are asked to prove instead of … that leads to the proof/ concept. Most learners cannot interpret more complex geometric diagrams* (E2).

The above responses suggest that learners' geometric difficulties increase with an increase in the complexity of geometrical diagram. In addition, L37, E2 and E4's responses suggest that learners' poor understanding of relationships among properties of shapes leads to the indiscriminate use of theorems. Some of the learners' interview responses indicate that they were taught to memorise theorems as they are in the textbook, with no practical experience of how they are derived. Thus, learners lacked understanding and active classroom involvement in the derivation of theorems from relationships among properties of shapes. Due to poor concept formation, they confuse statements and reasons. These learners attributed the challenges they have at informal deduction in geometry to poor teaching methods, which discouraged active learner participation in conceptual development.

5. Summary of findings and discussion

This article aimed to explore and describe the learning difficulties in Euclidean geometry at Van Hiele's first three levels of geometric thinking. Presented below is an overview of the findings of the study from the data gathered at different Van Hiele levels:

5.1 Study question 1: What are the learners' challenges pertaining to geometry related to Van Hiele's visualisation level?

Overall test results revealed that 68% of learner respondents passed at Van Heile's visualisation level, compared to only 38% who failed to get 50% at this level. These figures suggested that most Grade 12 learners have a good mastery of visualisation skills, with a few of them still lacking recognition and intuitive abilities to achieve at this level. Learners' lack of abilities stems from their challenges in recognising and naming angles within a geometrical diagram. Amongst plausible explanations for these results is that learners struggle to integrate the skills to visualise geometrical diagrams with their visual thinking capabilities, as evidenced in both the learners and the educators' interview responses. The findings were consistent with Mosia *et al*.'s (2023) study, which found that learners only have a partial understanding, often limited by visual reasoning of the theorem. Thus, the results suggest that some learners could not apply the mental images they had created from their previous experiences with geometric diagrams at visualisation level in to solve unfamiliar geometry problems. Therefore, there is a need for educators to implement the model of Van Hiele to teach geometry as a way to improve learners' visual thinking and spatial abilities.

5.2 Study question 2: What are Grade 12 learners' challenges pertaining to geometry in relation to the analysis/descriptive level?

Test results showed that 55% of the learners did not pass at 50% to questions set at the analysis level of Van Hiele's hierarchy of geometric thinking. These results showed that learners had difficulties in describing the components of a circle using the correct geometry terminology, even though they could identify and name the circle components. The results are consistent with those of Maqoqa (2024), who found that learners are often confused when naming and specifying properties of given geometric shapes. In addition, the results suggest that learners cannot use properties of a circle to write standard definitions for its components. In this vein, interview responses revealed that learners lack descriptive abilities, even though most of them could identify and name those components. Thus, some of the learners argued that these challenges are a result of the failure by their educators to involve them in discovering properties of shapes by themselves during teaching and learning. These findings support an earlier suggestion by De Villiers (1998) that indicates that educators should empower learners by engaging practically them in geometric activities rather than teaching the definition of geometric concepts. It therefore suggests that learners' challenges with the understanding and correct application of theorems in geometry problem-solving can be attributed to their lack of analytical skills.

5.3 Study question 3: What are the Grade 12 learners' challenges pertaining to geometry challenges in relation to the informal deduction level?

Test results revealed that 70% of the learners failed to establish relationships among properties of shapes within geometrical diagrams, especially those involving triangles and quadrilaterals. As a result, findings were that most learners could not write correct reasons to geometric statements. Furthermore, research results suggest that very few learners could successfully operate at levels beyond visualisation as they struggled to reason deductively from using property relationships. Similarly, at the informal deductive level most learners indicated that they rely more on intuition than logic as the basis for theorem application; hence they end up using theorems indiscriminately. This confirmed literature finding that most learners in South Africa remain fixed at concrete visualisation, whereas they are expected to operate at higher levels of geometric thinking (Alex & Mammen, 2016; 2018; Mosia et al., 2023). As a result, learners that did not achieve at this level wrote incorrect statements or reasons, or both. We also found that learners had difficulties in understanding the mathematical procedure of supporting a statement with a reason, suggesting that they lack confidence to use their reasoning skills.

Overall, findings were that learners tend to experience more serious challenges in interpreting more complex geometrical diagrams due to their failure to establish relationships among properties' shapes and make relevant deductions. The findings concur with Maqoqa (2024), who reveals that some of the learners are unable to identify the proper characteristics of geometric shapes. Therefore, conclusions were that learners' poorly developed logical thinking skills and misunderstanding of circle theorems lead to their failure to solve simple riders.

6. Conclusion

Based on the study outcomes, the non-achievement rate at visualisation level was 19%, indicating that some learners had difficulties in recognising and naming different shapes. Those learners could not establish how shapes are alike and different, and they lacked the gestalt-like approach to shape. We recommend that educators integrate the teaching of visualisation and visual thinking with the manipulation of real objects and shapes for them to develop good spatial skills. Similarly, findings suggest that most learners do not have the requisite vocabulary/language for geometry. As a result, learners' failure to master appropriate geometry terminology can hinder their understanding of concepts involved. That is evidenced by most learners' failure to use formal geometry terminology to describe the components of a circle using formal geometry terminology. Learners had difficulties classifying shapes according to their properties. Therefore, due to the identified learners' limited abilities in using geometry terminology to describe relationships within a geometric diagram, we recommend that educators teach level-specific vocabulary in line with Van Hiele's hierarchy to ensure that learners build mental images of geometric concepts and not rely on rote memorisation.

Furthermore, research findings revealed that most learners lacked both conceptual knowledge and procedural fluency to solve geometry problems successfully at Van Hiele's informal deduction level 2. That was demonstrated by most learners' inability to relate properties of geometric shapes, apply correct and relevant theorems as reasons for geometric statements when solving one-step geometric riders. Results further suggest that learners lacked logical arguments about the properties of shapes. We found that some learners could not follow and appreciate an informal deductive argument about shapes and their properties as well as using their intuition to establish those relationships. Therefore, geometry teaching and learning should emphasise learners' construction of knowledge of geometry and ways to overcome misconceptions and errors in geometry because they are the major causes for the learning challenges in Euclidean geometry (Mackle, 2017; Mosia et al., 2023). Consequently, we recommend that educators use Van Hiele's model when teaching geometry, as it focuses on the constructivist teaching approaches that emphasise conceptual understanding instead of rote memorisation of geometric facts.

Competing interests

The authors declare that they have no financial or personal relationships that may have inappropriately influenced them in writing this article.

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