The role of visualisation in developing critical thinking in mathematics

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Research has been conducted on the role and importance of visualisation in many fields, including psychology, but very little has been done to extend its role to mathematics education in particular. Furthermore, much research has been done on the importance of critical thinking. However, to date not much has been done to clarify the fact that visualisation is a very important component of critical thinking. In this paper I investigate visualisation for the purpose of examining its role in developing critical thinking for a better understanding of mathematics by Grade 9 learners. In order to achieve this, I provide a qualitative examination from a master's dissertation of some thought processes used during visualisation when Grade 9 pupils engage in mathematics tasks. In order to support the derived thought processes data, audio taped interviews were conducted with the students. A framework of four stages of visualisation was adapted for the collection and analysis of data. Several roles of visualisation were identified, which revealed that visualisation encourages critical thinking which further leads to a better understanding of data handling.

Keywords: critical thinking; mathematical understanding; mathematics; mental processes; visualisation

Introduction

Educators should teach learners how to think, but instead they frequently teach them what to think (Clement & Lochhead, 1980). Educators usually succeed in teaching learners the content of respective academic disciplines, but teaching learners how to think effectively about subject matter is often not achieved. Understanding how learners think during problem solving, in order to help them think effectively about mathematics, is complex and requires special training. Mathematical visualisation is the process of forming images or constructing mental representations and using such images effectively for mathematical discovery and understanding, whereas critical thinking is a mode of thinking about any subject content or problem in which the thinker improves the quality of his or her thinking by skilfully taking charge of the structures inherent in thinking and imposing intellectual standards upon them (Scriven & Paul, 2005:2).

Campbell, Watson and Collis (1995:177) suggest that the development of mathematically competent and critical learners involves not only the traditional focus on computational and logical problem solving abilities, but also the development of associated visual images and intuitive skills at all stages in the developmental process. Learners are not born with the power to think critically, nor do they develop this ability naturally. It is a learned ability, the development of which needs to be facilitated. Much of our thinking, left to itself is biased, distorted, partial, uninformed or downright prejudiced (Scriven, 2005). The quality of our life and what we produce, make or build, depends precisely on the quality of our thoughts. Since critical thinking should be prominent in the learning of mathematics, any strategy used to develop and encourage it is considered useful. In this paper I argue that all mathematics teaching should focus on "teaching for understanding", with visualisation as the important cornerstone. The key research question addressed here is: How do the processes of visualisation promote critical thinking? Therefore my aim is to determine how the process of forming images or constructing mental representations through visualisation enriches the quality of critical thinking by taking charge of the structures responsible for thinking. I discuss some of the findings from a dissertation (Makina, 2005), the aim of which was to determine the nature of the mental processes involved in problem solving during visualisation. The development of critical thinking through visualisation in the learning of mathematics was identified through these processes.

All the tasks were chosen from the authors Fischbein (1987) and Reading (1996), whose significant articles focused on improving visualisation in mathematics.

Visualisation

There is increasing recognition that visualisation plays a role in the learning of mathematics (Ben Chaim, Lappan & Houang, 1989; Davis, 1984; Presmeg, 1992; Thornton, 2000; Wheatley, 1991). As a result, researchers have contributed several useful ideas about visualisation. Kosslyn (1994:1 23) describes visualisation as a cognitive process involving visual imagery in which images are either generated, inspected, transformed or used for mathematical understanding (see Table 1). Visualisation incorporates those mental processes that make use of, or are characterised by, visual imagery, visual memory, visual processing, visual relationships, visual attention and visual imagination. Presmeg (1992:40) describes visualisation as an aid to understanding, and one can therefore speak about visualising a concept or a problem. To visualise a problem means to under stand the problem in terms of a visual (mental) image hence the visualisation process is one that involves visual imagery, with or without a diagram, as an essential part of the method of solution (Presmeg 1992:298). The term "visual" here refers to the manner in which mathematical information is presented and processed during or before problem solution. Presmeg (in Thornton, 2000:254) lists five different kinds of visual imagery she identified in her learners: concrete, pictorial imagery (pictures in the mind); pattern imagery (pure relationships depicted in a visual spatial scheme); memory images of formulae; kinaesthetic imagery (involving muscular activity); and dynamic (moving) imagery. The range of visualisations generated by individuals is therefore an important factor to keep in mind when considering mathematics teaching. On the other hand, Cobb, Yackel and Wood (1988:26 27) regard visualisation as a dualism created between mathematics in learners' minds and mathematics in their environment, present in what they call "the representational view of mind", which they find to be prevalent in mathematics education today. In the representational view of mind, the overall goal of instruction is to help learners construct mental representations that correctly or accurately mirror mathematical relationships located outside the mind in instructional representations.

The descriptions offered by Kosslyn (1994) and Presmeg (in Thornton, 2000) are central to this paper. They all characterise visualisation as mental processes or schemas that can result in a recognised product (visual image). As learners gain experience of life in general, and mathematics in particular, the constraints on their mental activity change. They can develop and become more versatile, they can decay, or they can change by conscious and unconscious reformulations of ideas as the learner attempts to make a coherent pattern out of the universe in which he or she lives. Understanding that comes about through this search for coherence is termed "relational under standing" (Tall, 2008). These schemas or mental processes and images are in the mind and cannot be seen, but have to be understood through the study of the mind. Visualisation offers a method of seeing the unseen and we are encouraged and should aspire to 'see' not only what comes 'within sight', but also what we are unable to see (Arcavi, 2003:215). This stands as the starting point in describing visualisation as an important aspect of mathematical understanding, insight and rea soning, which, in turn, enhances the learner's critical thinking.

The findings of the different authors cited above have very important implications for mathe matics education and serve as integral parts of teaching. They serve as important aspects of mathe matical understanding, insight and reasoning. Visualisation helps teachers to make instructional decisions about how to teach, the content and the nature of tasks. Furthermore, it aids teachers with the facilitation of lessons and with the ability to engage learners in realistic situations. For example, an understanding of the precise limitations of children's imagery could indicate how to use imagery in teaching. If learners' images are static, in that learners cannot transform their images (see them being moulded into a new shape), they may still be able to make use of "blink" transformations, i.e. erasing the first image and imaging the object anew in some altered way (Bishop, 1989:8). If so, it makes sense to shift from trying to teach learners the rules of transformation to teaching them the rules of formation. Learners need guidance in order to engage in reflective and independent thinking so as to justify and reflect on their values and decisions.

For the last century, teachers of mathematics have been attempting to replace the teaching of critical thinking in favour of establishing environments that allow for critical thinking that is possible through discussion and interaction. The National Council of Teachers of Mathematics (NCTM) in the USA echoed a call for classrooms that place critical thinking at the heart of instruction and sig nalled that a pervasive emphasis on reasoning would be an essential aspect of all mathematical activity (National Council of Teachers of Mathematics, 1989). This takes us one step further in questioning our assumptions about the meaning of critical thinking. Students, as human beings, are critical thinkers, and they would display these skills if the classroom allowed such behaviour (Skovsmose, 1994), i.e. the behaviour promoted through visualisation.

Teaching for understanding

Primarily, visualisation should be the key to successful learning and understanding. Mathematical thinking and understanding, according to Duval's framework (2000), requires the co ordination of at least two registers, one is multi functional (open to multiple interpretations in verbal and geometric registers) and the other is mono functional (open to one interpretation in symbolic and graphical registers). Tall (2008) analyses how the development of the understanding of objects and processes interact. Mathematical understanding is achieved as learners learn to discriminate between and coordinate semiotic systems of representation. The semiotic systems of representation, or regis ters, refer to four classifications of systems of representation: generation, inspection, transformation and use (Kosslyn, 1994). Some learners' difficulties in the construction of concepts are linked to the restriction of representations in their learning experiences. Teachers of mathematics have been searching for ways to describe and enact critical thinking in their classrooms for a very long time. On the one hand, mathematics itself is often held up as the model of a discipline based on rational thought, clear concise language and attention to the assumptions and decision making techniques that are used to draw conclusions. Students, however, learn mathematics through experiences of critical thinking (Fawcett, 1995). Visualisation is one way in which experiences of critical thinking are gained. We should take advantage of the critical thinking skills that students bring with them to school mathematics in order to learn mathematics. Fawcett suggests ways that students could demonstrate that they are, in fact, thinking critically as they participate in the experiences of mathematics in the classroom: selecting significant words and phrases in any important statement and defining them; supplying evidence to support conclusions they are pressed to accept; analysing that evidence and distinguishing fact from assumption; recognising stated and unstated assumptions essential to the conclusion; evaluating these assumptions, accepting some and rejecting others; evaluating the argument, accepting or rejecting the conclusion; and constantly re examining the assumptions that are behind their beliefs and actions (Fawcett, 1995).

Critical thinking

Critical thinking is a very powerful way of thinking in every sector of one's life. Van den Berg (2004:279) comments that in an increasingly complex and specialised society, it is imperative that individuals think critically. This kind of thinking is required to achieve the critical outcomes stated in the National Curriculum Statement for South Africa (Department of Education, 2004). Generally, critical thinking means thinking in pursuit of relevant and reliable knowledge about a subject or the world (Schafersman, 1991:3). It is reasonable, reflective, responsible and skilful thinking that is focused on deciding what to believe or do. Critical thinking is a mode of thinking about any subject content or problem in which the thinker improves the quality of his or her thinking by skilfully taking charge of the structures inherent in thinking and imposing intellectual standards upon them (Scriven & Paul, 2005:2). To arrive at a creative solution to a problem involves not just having new ideas, but that the newly generated ideas should be useful and relevant to the task at hand. Critical thinking plays a crucial role in evaluating new ideas, selecting the best ones and modifying them, if necessary, by providing the tools for the process of self evaluation. This means having the ability to engage in reflective and independent thinking. In this way critical thinking emphasises the skills of analysis, teaches learners how to understand, follow or create a logical argument, work out the answer, eliminate the incorrect paths and focus on the correct one (Harris, 2002:1).

In mathematics, critical thinking is thinking about what is being asked in a given problem, determining what operations and procedures are used in a mathematics problem, with help from a mathematics teacher, and sharpening the analytical skills of learners to improve their mathematics (Chang in Appelbaum, 2004). Someone with critical thinking skills in mathematics is able to understand the logical connections between ideas; identify, construct and evaluate arguments; detect inconsistencies and common mistakes in reasoning; solve problems systematically; identify the relevance and importance of ideas; and reflect on the justification of his or her own beliefs and values (Clement & Lochhead, 1980). Scriven and Paul (2005:2) comment that a well cultivated critical thinker raises vital questions and problems, formulates them clearly and precisely; gathers and assesses relevant information and uses abstract ideas to interpret them effectively; comes to well reasoned conclusions and solutions and tests them against relevant criteria and standards; thinks open mindedly within alternative systems of thought and recognises and assesses as need be, their assumptions, and communicates effectively with others to find solutions to complex problems. Critical thinkers gather information from all senses: verbal and/or written expressions, reflection, observation, experience and reasoning. It is therefore important to enable one to analyse, evaluate, explain and restructure one's thinking, thereby decreasing the risk of acting on or thinking with a false premise. Nickerson (1987) characterises a good critical thinker in terms of knowledge, abili ties, attitudes and habitual ways of behaving. A person who thinks critically can ask appropriate questions, gather relevant information efficiently, sort through this information creatively and come to reliable and trustworthy conclusions.

According to Schafersman (1991), the key to appreciating the significance of critical thinking in the mathematics classroom can be found in two phases. The first one occurs when learners (for the first time) construct in their minds the basic ideas, principles and theories that are inherent in the content during visualisation. This is a process of internalisation. The second occurs when learners use those ideas, principles and theories effectively as they become relevant in learners' minds during the process of application. Good critical thinking can therefore be seen as the foundation of mathematics since it enhances general thinking skills, promotes creativity and is crucial for self reflection. The key issue here is that the teacher who fosters critical thinking fosters reflectivity and visualisation in learners by asking questions that stimulate thinking, which is essential to the construction of knowledge. The above discussion implies that good critical thinking in mathematics will not occur unless learners are engaged in activities that deliberately promote this kind of thinking. These results can be achieved when visualisation is utilised in the learning of mathematics and the correct motivation and attitude prevail throughout persistent practice,.

Methodology

Twelve Grade 9 learners were purposefully selected using a non probability sampling procedure in collaboration with the mathematics teacher at a secondary school (Makina, 2005). The distribution of the 12 learners was as follows: four above average learners; four average learners; and four below average learners, selected on the basis of previous class performance. All data were collected during organised one hour sessions in which learners gave written responses based on the tasks (see Appendix A), followed by individual audio taped interviews to support data from the written responses. The tasks chosen were selected because they had been used previously in other research (Fischbein, 1987; Reading, 1996) to improve visualisation and could be easily adapted for use in mathematics classrooms. Kosslyn (1994) suggests four stages and processes in the tasks that prove that visualisation has taken place. The analysis of both the written responses and the interviews were aided by Kosslyn's (1994) four stages of visualisation: image generation, image inspection, image transformation and image use, as illustrated in Table 1.

Data analysis, results and discussions

The investigation revealed some major processes, namely, reflection, self regulation, divergent thinking, meta cognition, mathematical generalisation, transformation, the use of metaphors, similes, and animation (Makina, 2005), which could result in critical thinking. All these mental processes are interrelated and emerged as a result of the fact that learners described what they were thinking

Table 1. Description of Kosslyn's (1994) categories of mental processes used for analysing data

Process	Description of process
Image generation	• Occurs when a person recalls a picture or visual mental representation from long term memory and places that image in a central location within working memory. • Makes multiple use of materials (incorporating new experiences).
Image inspection	• Involves a focused mental scanning of the qualities of the image within working memory. • Results when one examines an image in order to answer questions about it (classify, scrutinise). • Entails noticing similarities and differences in shapes (connection). • Selectively attending to input and feedback from existing schemata (heuristic processes); establishing meaning of the problem developing tactics and self monitoring.
Image transformation	• Occurs when a person changes or operates upon an image: rotating, zooming, translating, scanning. • Involves the movement of images of shapes into related shapes (e.g. square to hexagon).
Image use	• Occurs when an image is employed in the service of some mental operation. • It is the mental involvement in making the shape (refinement). • Involves making, using, changing and storing images, concepts, understandings and schemata.

about while working through mathematics tasks. In accordance with the literature, learners were able to 'see the unseen' (mental images) in their minds where all the visualisation processes took place. For example, one learner gave the following answer regarding Task 1: "*I am trying to make a shape and I am thinking of a shape in my head. I am thinking of the dice*". The images that resulted from the 'seeing' were only partly presented on paper. The conclusion is that the whole process of visualisation can never be fully illustrated on paper and the teacher has to probe orally to ascertain most processes.

The earlier section on critical thinking clarified the characteristics and expectations of a critical thinker. The mental processes identified during visualisation while problem solving were in accord ance with critical thinking as described in the literature study. These expectations were related to problem solving processes, which, in turn, were also related to Kosslyn's (1994) processes of visualisation. In this research, most processes that were dominant during visualisation while problem solving, are related to most of the processes expected of a critical thinker (Alsorour in Harris, 2002:227; Harris, 2002:226). Example 1 and Figure 1 show evidence of generation, inspection and, finally, the transformation that occurred during the execution of Task 1. A geometric body was described in the way that each learner understood it from previous knowledge, as shown in Example 1. This is divergent thinking and it derives from learners' higher order thinking skills.

Example 1

LEARNER I: "*A geometric body has air in it, so I made a box*".
LEARNER A: "*A geometric body has got sides all equal like it can form something …*"
LEARNER C: "*I discovered that a geometric body has a … should have space inside …*"

Learner B's written work, depicted in Figure 1, shows a learner progressing from one shape to another, and is evidence of an original image that was generated and then went through a number of transformations after inspection. In some cases, the cube was named a toy box or a box for a jack in the box.

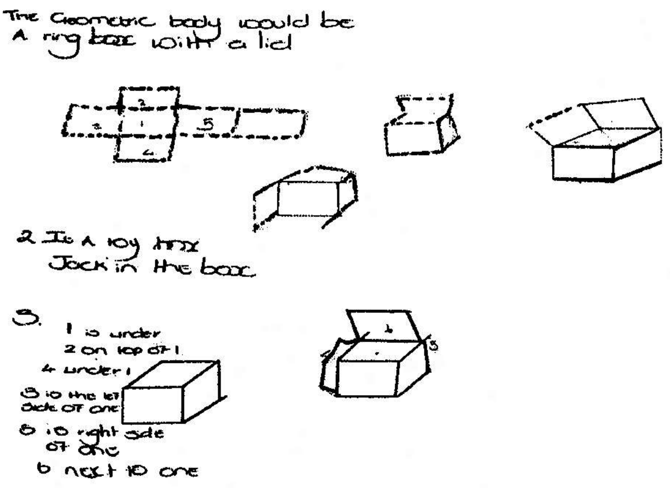


Figure 1. Learner B's written response to Task 1

Self monitoring is identified in the above figure by means of the draft plans/trial diagrams of the box. There are also several descriptions of what the student thought was an appropriate name/ label for this figure. The self monitoring process on the part of the learners seems to play a role in their successful completion of the tasks. Visualisation therefore appears to allow self monitoring during the problem solving processes. Van den Berg (2004:292) states that self assessment or self monitoring facilitates critical thinking processes by influencing learners' responsiveness and thus plays a key role in determining how learners selectively paid attention during problem solving. During visualisation, creativity, metaphors, similes and animation are used to guide the reasoning process (Presmeg, 1992:599). Task 1 required the learner to draw on paper the shape that could be made from the net shape that was given. Learner I made an actual cube out of paper, glue and Sello tape to clarify for herself the shape that was made from the net shape. During the interview, she explained the following:

INTERVIEWER: "You made a box with the sides 2cm each"?

LEARNER I: "*I thought maybe I was just supposed to identify what this thing could make*".

INTERVIEWER: "So you didn't want to draw anything on paper? It's like the problem says: 'On the plain piece of paper provided, draw the image obtained by unfolding or folding the geometric body".

LEARNER I: "*Ja. It's like to me it was obvious that I needed to make it. When I looked at the paper I could see that in this thing we could do something like this box which can be six sides which is 1, 2, 3, 4, 5, 6. So I thought maybe I could make this box*" (showing the researcher the box she had physically made from paper, glue and Sellotape).

The above conversation provides evidence that learners at times were not always able to draw or write what they could mentally imagine. In substantiating this, Wheatley (1998:10) maintains that care must be exercised in inferring imaging from children's drawings, as learners cannot always draw what they can imagine. It is therefore true that the mental images of some learners have to work hand in hand with physical visual images if the learners are to be better critical thinkers. Visualisation facilitates the use of concrete objects as mental visual aids during problem solving.

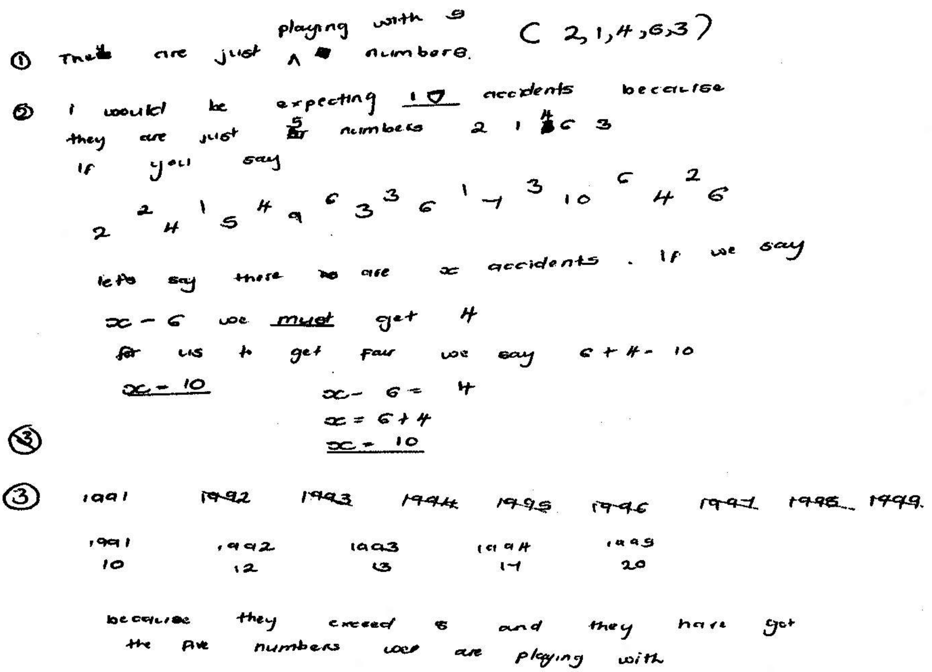


Figure 2. Learner E's written response to Task 2

It must, however, be noted that concrete materials are not used to transmit visual images or spatial concepts to learners, but are vehicles to encourage problem solving, knowledge construction and the development and use of different kinds of imagery (Vinner in Owens & Clements, 1998: 215).

Learner E's response shows the learner trying to make sense of the numbers as well as the pattern by inserting extra numbers, some calculations and an explanation. In Figure 2, learner E developed her understanding of the problem, demonstrated increasingly abstract solution activity so that she could anticipate its results, and was able to use the results to make future predictions through generalisation. Visualisation generated guided tactics, manipulations and decisions during problem solving. Therefore, it plays a central role in inspiring a whole solution beyond the merely procedural, towards a more organised structure (Arcavi, 2003:216 233).

Task 2 allowed the learners to use the results of their answers to part one to make predictions in parts 2 and 3. Learners approximated the number of accidents after 1991 by studying the relationships that were identified in the first part of the question.

Example 2

RESEARCHER: "How did you predict the number of accidents that occurred in 1992"?

LEARNER F: "*I could see it by looking at these numbers. There was a pattern that I recognised and if this pattern is to continue, 1991 will get about 10 accidents …*"

In Example 2, Learner F recognised the need to use the word "about" to provide an answer to the number of accidents that occurred in 1991. Visualisation through higher order thinking skills offers a method of estimation and therefore enables prediction of results in data handling. Critical thinking during visualisation reduces the learner's mental limitations in thinking, learning and problem solving activities. This was seen in this study when the learners were involved in solving problems that involved patterns that were too long to determine the *n*th term, or when they needed to make future or previous predictions. Visualisation in the service of problem solving may also play a central role in inspiring a whole solution. For example, when required to make a generalisation from a pattern of numbers or shapes, visualisation identifies gnomes as substructures of the whole in which a clear pattern can be established (Ben Chaim *et al.*, 1989).

Conclusion

Arcavi (2003), Reading (2005), Fischbein (1987), Ben Chaim, Lappan & Houang (1989), Wheatley (1991), Davis (1984), Presmeg (1992) and Thornton (2000) have all, at one point, highlighted the important processes involved in visualisation when learning mathematics. There has, however, not been a study of the important relationship that exists between visualisation and critical thinking, and more research should be done in this area. Visualisation is a very important component of understanding, and critical thinking determines the quality of the understanding. Regrettably, most learners do not develop appropriate thinking skills that enable them to make meaningful theoretical distinctions, because some teachers tend to focus only on the task of transmitting basic knowledge. It is counterproductive in mathematics education simply to require learners to memorise information and to learn new and isolated facts without understanding why they think the way they do. Without critical thinking, which is self directed, self corrective thinking the learning of mathematics is difficult, as it relies on the memorisation of knowledge. Visualisation processes must occur for a person to be able to think critically. With visualisation, the overall goal of instruction is to help learners construct mental representations that correctly or accurately mirror mathematical relationships in instructional representations located outside the mind.

The challenge in mathematics teaching is how to help learners to construct appropriately linked cognitive units that are flexible and precise, to help them build mathematics as a coherent and meaningful structure. This can be achieved by engaging learners in problem solving, where problems and ideas are discussed while probing learners' thinking through questions and conflicting situations. Learners also need to analyse their own thinking while fostering the gradual emergence of all their mental processes. Encouraging learners to reveal their mental processes can be successful if educators have a clear understanding of the role of visualisation. The following suggestions by Thornton (2000:255) are a starting point for teachers to help learners become more effective visual thinkers and more sensitive to the possibility of finding visual solutions or representations of a given result, while encouraging concrete pictorial imagery by asking learners to picture themselves as part of the situation. This promotes the discussion of alternative ways of thinking, particularly of the transition from visual to symbolic thinking, and encourages learners to view problems holistically, instead of breaking them into parts.

Finally, the investigation in this paper, which resulted from a larger study (Makina, 2005), has revealed a number of major processes, namely reflection, self regulation, divergent thinking, metacognition, mathematical generalisation, transformation and the use of metaphors, similes and animation. These are some of the main processes that enrich the quality of thinking when learners learn, hence promoting critical thinking. Complex ideas, which are a result of critical thinking, are among the potential products of visualisation. Visualisation is therefore central to the development of critical thinking processes and the construction of meaning and understanding. Understanding the learners' thinking during problem solving (visualisation) can enhance the way teachers develop critical thinking in learners. It can be concluded from this paper that learners are likely to improve their performance in mathematics if teachers remain aware of visualisation, which is one of the important processes through which to encourage critical thinking in the classroom.

References

Appelbaum P 1995. *Popular culture, educational discourse and mathematics*. Albany, NY: State University of New York Press.

Appelbaum P 2004. *Critical thinking and learning: An encyclopaedia for parents and teachers.* J Kincheloe & D Weil (eds). Greenwood Press, Mathematics Education Arcadia University.

Arcavi A 2003. The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, 52:215 241.

Ben Chaim D, Lappan G & Houang RT 1989. The role of visualisation in the middle school mathematics curriculum. *Focus on Learning Problems in Mathematics*, 11:49 60.

Bishop AJ 1989. Review of research on visualisation in mathematics education. *Focus on Learning Problems in Mathematics*, 11:7 16.

Campbell KJ, Watson JM & Collis KF 1995. Visual processing during mathematical problem solving. *Educational Studies in Mathematics*, 28:177 194.

Clement J & Lochhead J 1980. *Cognitive process instruction*. Cognitive Development Project & Department of Computer & Information Science, Amherst, MA. Philadelphia: Franklin Institute Press.

Cobb P, Yackel E & Wood T 1988. *Curriculum and teacher development as the co ordination of psychological and anthropological perspectives*.

Davis RB 1984. *Learning mathematics: The cognitive science approach to mathematics education*. Routledge, London.

Department of Education 2004. *National Curriculum Statement Grades 10 12* (*General*): *Mathematics*. Pretoria: Government Printer. Retrieved from http://www.thutong.org.za/ on 2 January 2008.

Duval R 2000. *Basic issues for research in mathematics education*. Proceedings of the Conference of the International Group for the Psychology of Mathematics Education (PME 24), 23 27 July, Hiroshima, Japan.

Fawcett H 1995. *The nature of proof* (*National Council of Teachers of Mathematics (NCTM) 1938 Yearbook*). Reston, VA: National Council of Teachers of Mathematics.

Fischbein E 1987. *The nature and role of visualisation and imagery in the teaching of mathematics 1989*. Retrieved from http:// www.soton.ac.uk/~dkj/bsrlmgeom/ orts/K Jones et al June 1998.pdf on 15 February 2004.

Harris R 2002. *Creative thinking techniques*. Retrieved from http://www.virtualsalt.com/crebook2.htm on 20 November 2007.

Kosslyn MS 1994. *Image and brain: The resolution of the imagery debate*. London: W.W. Norton.

Makina A 2005. *An investigation of the role of visualisation in data handling in Grade 9 within a problem centred context*. Unpublished dissertation, University of South Africa.

Mgqwashu EM 2009. Rethinking academic literacy for educators: towards a relevant pedagogy. *Perspectives in Education*, 27:215 227.

Mudaly V 2009. Iterative processes in mathematics education. *Perspectives in Education*, 27:302 310.

National Council of Teachers of Mathematics (NCTM) 2000. *Principles and standards for school mathematics*. Reston, VA. Retrieved from: http://standards.nctm.org/ on 7 February 2009.

National Council of Teachers of Mathematics (NCTM) 1989. *Curriculum and evaluation standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Owens KD & Clements MAK 1998. Representations in spatial problem solving in the classroom. *Journal of Mathematical Behaviour*, 17:97 218.

Presmeg NC 1992. Prototypes, metaphors, metonymies and imaginative rationality in high school mathematics. *Educational Studies in Mathematics,* 23:595 610.

Presmeg NC 1997. *Generalization using imagery in mathematics, in Mathematical reasoning, analogies, metaphors and images*. L English (ed.). Mahwah: Erlbaum

Ramranthan P & Amin N 2009. Preparing students to teach in and for diverse contexts: A learning to teach approach. *Perspectives in Education*, 27:69 77.

Reading C 1996. *An investigation of students' understanding of statistics*. Unpublished thesis, University of New England, Armidale, Australia.

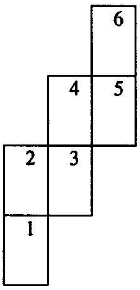
Schafersman SD 1991. *An introduction to critical thinking*. Retrieved from http://www.freeinquiry.com/ critical thinking.html on 16 January 2008.

Scriven M & Paul R 2005. The critical thinking community. Retrieved from: http://www. Critical thinking.org.about CT/define critical thinking.cfm on 15 January 2008.

Skovsmose O 2000. Aphorism and critical mathematical education. *For the Learning of Mathematics,* 20:2 8.
Skovsmose O 1994. *Toward a philosophy of critical mathematics education*. Dordrecht, The Netherlands: D. Reidel.
Tall D 2008. *The dynamics of understanding mathematics*. Warwick: University of Warwick.
Thornton S 2000. *A picture is worth a thousand words*. University of Canberra. Retrieved from http://www .amt.Canberra.edu.au/ sjt/dva.htm on 7 February 2003.
Wheatley GH 1991. *Enhancing mathematics learning through imagery*. Retrieved from www.aare.edu.au/92pap/collk92019.txt 27k on 2 April 2004.
Van den Berg G 2004. The use of assessment in the development of higher order thinking skills. *Africa Education Review,* 1:279 294.

Appendix A

Task 1



1. Identify the geometric body or image which could be obtained by imagining the folding back of the multi-dimensional drawing shown above.
2. On the plane piece of paper provided draw the image obtained by unfolding or folding the geometric body (actually perceived or mentally represented) by the figure above.
3. Clarify your diagram by indicating the number which will be opposite the other.

(Adapted from Fischbein(1987))

Task 2

Data Interpretation Question

A well-known intersection in Armidale has had a number of serious accidents. The number of serious accidents was recorded for the last ten years.

1981	1982	1983	1984	1985	1986	1987	1988	1989	1990
2	4	5	9	3	6	7	10	4	6

1. Describe any pattern that you can see in the data.
2. Approximately how many accidents would you expect in 1991? Why?
3. Suggest four other years in the future (after 1990) when you think the number of accidents might exceed 8. Why did you select those years?

(Adapted from Reading(1996) University of New England, Australia)