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First-year university students' understanding of function concepts encountered in Grade 12 Mathematics: A case study

Abstract

This article focuses on first-year university students' understanding of concepts related to third-degree polynomial and trigonometric functions that they encountered during their study of Grade 12 mathematics. In this study three online questions from two first-year mathematics quizzes at the University of KwaZulu-Natal were analysed. The first question focused on the characteristics of a polynomial function, while the second and third questions focused on the characteristics of trigonometric functions. The characteristics included calculus-related concepts, for example, intervals of increase or decrease, concavity and local extrema. It was found that about a fifth of the participants (n = 557), science students who studied the core mathematics module Introduction to Calculus*, had difficulties in answering those calculus-related questions compared to determining the general non-calculus characteristics, for example, the range and domain. The study also found that about a quarter of the participants lacked relational understanding with regard to calculus-related concepts. This study recommends that lecturers need to spend more time on calculus-related concepts of a function that focus on relational understanding.*

Keywords: *calculus concepts, instrumental and relational understanding, polynomial function, trigonometric function,*

1. Introduction

The prescribed textbook for first-year university mathematics (Stewart, Clegg & Watson, 2021) defines a function as follows: A function *f* is a rule that assigns to each element *x* in a set *D* exactly one element, called *f(x)*, in a set *E*. In South Africa, functions are part of the secondary schooling mathematics' syllabus. The function concepts are a prerequisite for a study of calculus (Maharaj, 2013), both at Grade 12 level and at university level. This implies that it is important to determine students' understanding of the concepts related to functions before they proceed with their studies in calculus. This case study focused on the level of understanding displayed by incoming first-year university mathematics' students in the context of concepts related to third-degree polynomial and trigonometric functions that they covered in their secondary schooling years.

The polynomial function is examined in *Mathematics Paper 1* of the Grade 12 examination, while trigonometric functions are examined in *Paper 2*. Each of these sections contribute to a weighting of about 8% of the marks in each of the examination papers. Table 1 indicates that sections on functions and their graphs are an important part of the end-of-year papers; especially in *Paper 1;* in *Paper 2*, the assessing is done as a part of trigonometry.

Table 1: Mark distribution for Mathematics NCS end-of-year papers: Grades 10-12

PAPER 1: Grade 12: bookwork: maximum 6 marks			
Description	**Grade 10**	**Grade 11**	**Grade 12**
Algebra and equations (and inequalities)	30 ± 3	45 ± 3	25 ± 3
Patterns and sequences	15 ± 3	25 ± 3	25 ± 3
Finance and growth	10 ± 3		
Finance, growth and decay		15 ± 3	15 ± 3
Functions and graphs	30 ± 3	45 ± 3	35 ± 3
Differential calculus			35 ± 3
Probability	15 ± 3	20 ± 3	15 ± 3
Total	**100**	**150**	**150**
PAPER 2: Grades 11 and 12: theorems and/or trigonometric proofs: maximum 12 marks			
Description	**Grade 10**	**Grade 11**	**Grade 12**
Statistics	15 ± 3	20 ± 3	20 ± 3
Analytical Geometry	15 ± 3	30 ± 3	40 ± 3
Trigonometry	40 ± 3	50 ± 3	40 ± 3
Euclidean Geometry and Measurement	30 ± 3	50 ± 3	50 ± 3
Total	**100**	**150**	**150**

***Source: Curriculum and Assessment Policy Statement Grades 10-12 Mathematics (DoBE, 2011)**

For this study, the results of three questions were analysed. Those questions were from two quizzes on functions which were created on the Moodle website for first-year mathematics students at the University of KwaZulu-Natal. The quizzes contributed to the calculation of the formative assessment mark of the students. When a student submitted an answer to a question, feedback was provided immediately by the system so that the student could determine his or her strengths and weaknesses, if any. This study focused on three major questions, one on a 3rd-degree polynomial function and two on trigonometric functions. Each of those questions contained a number of sub-questions based on concepts related to the study of functions. Those questions were chosen, since the preliminary analysis revealed that about a fifth of the students who attempted them had difficulties with the relevant concepts.

2. Research question

The main research question was: ***What is the level of university students' understanding of non-calculus and calculus related functions' concepts they encountered in Grade 12 mathematics?*** Some examples of non-calculus-related functions' concepts are domain, range, and *x*- or *y*-intercepts. Local extrema (maximum or minimum) and intervals of concavity are examples of calculus-related functions' concepts.

To help answer this question, concepts relevant to a 3rd-degree polynomial function and trigonometric functions were focused on.

3. Literature review

The diagnostic reports (Department of Basic Education, 2014) for mathematics based on the examination papers (Department of Basic Education, 2013a; 2013b) for the 2013 National Senior Certificate point out the performance of matric students under each section. The project on which this case study was based, began in 2014 and is ongoing. At that time, those were the latest diagnostic reports. In the diagnostic reports for 2019 and 2020 (Department of Basic Education, 2019, 2020), findings on the performance of students under each section are also indicated. Table 2 summarises from the relevant diagnostic reports the average performance of pupils for questions on the quadratic, cubic and trigonometric functions, and their graphs. One of the trends from this table is that students consistently scored below 40% for the sections on cubic functions and their graphs, and also around 30% for trigonometric functions and their graphs.

Table 2: Matric pupils' average performance on functions in percentages for 2013, 2019 and 2020

Questions on	2013	2019	2020
Quadratic function and graphs	68	55	48
Cubic function and graphs	20.6	39	39
Graphs of trigonometric functions	34.8	30	28

An analysis of the diagnostics reports (Department of Basic Education, 2014; 2019; 2020) revealed the common errors and misconceptions made by students for each of those sections indicated in Table 2. These could be summarised as follows:

Quadratic Functions: unable to plot points on the Cartesian plane; confused the quadratic function with the cubic function.

Cubic Functions: confused the finding of the turning point with the procedure (setting $f(x) = 0$) for finding *x* – the intercepts of the graph representing the function; concepts based on transformation of a function were poorly understood when used in the context of a given function.

The common errors or misconceptions for trigonometric functions are explained in the context of the Extract 1, Question 12 taken from the 2013 *Paper 2* examinations (Department of Basic Education, 2013b).

Trigonometric Functions: failure to detect the vertical translation involved to obtain the graph defined by $f(x) = \tan x + 1$ from the basic graph (see Extract 1, 12.1); confusing the domain of a function with its period (see 12.2); failure to describe the relevant transformations and reflection (see 12.3); inability to interpret *f'(x)* or write intervals correctly (see 12.4).

Extract 1: Question 12 from 2013 examination paper

QUESTION 12

12.1	Draw the graphs of $f(x) = \tan x + 1$ and $g(x) = \cos 2x$ for $x \in [-180° ; 180°]$ on the same system of axes provided on DIAGRAM SHEET 2. Clearly show all intercepts with the axes, turning points and asymptotes.	(6)
12.2	Write down the period of g.	(1)
12.3	If $h(x) = -\cos 2(x + 10°)$, describe fully, in words, the transformation from g to h.	(2)
12.4	For which values of x, where $x > 0$, will $f'(x)g(x) > 0$?	(4)
		[13]

Furthermore, the diagnostic reports for 2019 and 2020 also drew attention to the following: (1) Students lacked the understanding on how transformations influenced the equation of a graph. Therefore, there was a need for illustrations on how to develop a good understanding of how the graph changes when the equation changes and vice versa. (2) Some candidates were unable to read off the critical values correctly and consequently were unable to answer when a graph *f* was increasing or decreasing, or intervals of concavity. (3) Many candidates were confused between domain and range of a function. (4) When it was evident that candidates knew the interval(s) for which the graph was increasing/decreasing or the concavity of the function, they were, however, unable to write this interval using the correct notation. The four shortcomings of students implied in those diagnostic reports for 2019 and 2020 seem to be a result of what Skemp (1976) refers to as instrumental mathematics teaching. Such teaching emphasizes the knowledge of mathematical rules and how to apply them to get correct answers. Relational mathematics teaching is much deeper, in the sense that it requires one to know both what to do and why (Skemp, 1976). Skemp points out the following four advantages of relational mathematics teaching: *It is more adaptable to new tasks,* in the sense that relational understanding requires one to relate a particular method to a problem and even to adapt the method to new problem situations. Furthermore, if one knows how mathematical rules are interrelated as separate parts of a connected whole, then those mathematical rules are *easier to remember*; even though the initial learning may take longer to learn; once learnt it is likely to be more lasting. Relational understanding also lends itself to the teaching of more mathematical content. The ideas required for understanding a particular topic could turn out to be basic knowledge to understanding other topics in mathematics. For example, the basic concepts of sets and mappings are required for the understanding of the function concept. This lends itself to the motivational aspects for focusing on *relational knowledge,* since it could *be effective as a goal in itself*. Lastly, *relational schemas* (mind maps) *are organic in quality,* since they act as an agent of their own growth. This gives us the connection that relational understanding could be effective as a teaching goal in itself. If students get satisfaction from relational understanding, then not only will they try to understand relationally new material they are exposed to, but they could also actively seek out new material and explore new areas of knowledge.

The errors and misconceptions indicated in the diagnostic reports for 2019 and 2020, clearly show that most high school students do not know all the concepts required to understand polynomial and trigonometric functions, which makes it difficult for them to cope with more advanced functions at tertiary level. One would expect them to know how to draw and interpret quadratic, cubic and trigonometric functions by the time they have completed Grade 12 mathematics.

Many studies were conducted on students understanding of polynomial and trigonometric functions. Orhun (2001) found that many students have difficulties understanding the concept of the domain of a trigonometric function. In his study he took 77 students and asked basic trigonometric questions. One of the questions was: *What is the maximal domain of the function f(x) = sin x?* Only 9 (about 11%) wrote the correct answer. This underlines the lack of basic understanding of trigonometric functions in first-year students coming to universities. Orhun (2001) suggests that trigonometric functions and their related concepts should be taught by using their graphs.

Maharaj (2013) notes that students should have adequately established algebraic manipulation skills relevant to the concepts of a function before the concept of the derivative of a function is introduced. Commenting on the level of mental structures that students should have with regard to the APOS (action-process-object-schema) levels proposed by (Dubinsky, 2010) he states that students' mental structures of function should be at higher levels of APOS; process level and higher. Anabousy *et al.* (2014) found that the students had relative difficulty when treating the cubic function. They state that this difficulty is attributed to the complexity of mental construction needed to process the reflection of the cubic function about an axis, for example the *x*-axis. The difficulty also appeared in students' attempt to recognise the algebraic meaning of the reflection transformation. The insights from the relevant literature review were noted. Those insights were used in the design of the questions that the participants in this study had to respond to.

4. Conceptual framework

Skemp (1976) writes on understanding in general and identified understanding as either instrumental or relational. *Instrumental understanding* refers to the use of rules without understanding, while *relational understanding* refers to knowing what to do and why. The article by Dunnigan (2010) also states that there are generally two approaches to understanding: instrumental understanding and relational understanding. *Instrumental understanding* is described as having a mathematical rule and being able to use and manipulate it; knowing how to use that rule, but not knowing why it works. *Relational understanding* is described as having a mathematical rule, knowing how to use it and knowing why it works. Both types of understanding give the correct answer, but relational understanding is more extensive. In the context of extending one's mathematical knowledge and abilities, relational understanding is more useful in the sense that it is more adaptable to new tasks (Skemp, 1976). If this is accepted, then the ability to adapt one's knowledge and skills to solve problems in mathematics, and also in general, should be an important outcome of the learning and teaching situations. For relational understanding, mental structures in the context of APOS should be at the higher levels, process level and higher.

In the context of Question 12 (see Extract 1) given in the literature review, a student with instrumental understanding will be able to draw the graph of *y = tan x*, but could fail to use that graph to arrive at the graphical representation of the graph of $y = \tan x + 1$. If relational understanding with regard to translations and reflections of the graph of a standard function is undeveloped, then such a student is unlikely to detect from the structure of the defining equation $y = \tan x + 1$, the translation that is involved. For the example under discussion such a student will not detect that the graph of $y = \tan x$ is shifted vertically upward by one unit to arrive at the graph of the function defined by $y = \tan x +1$. Students with relational understanding (Skemp, 1976; Dunnigan, 2010) will be able to recognise that in the graph of a standard function defined by $k(x) = \cos x$; when $k(x)$ is translated by 10° to the left, the new graph $g(x) = \cos (x + 10^\circ)$ is formed, if ($x + 10^\circ$) is multiplied by 2 to produce $f(x) = \cos[2(x + 10^\circ)]$, the period is halved and in order to arrive at $h(x) = -\cos [2(x + 10^\circ)]$, $f(x)$ should be reflected about the *x*-axis. Furthermore, those with instrumental understanding (Skemp 1976, Dunnigan 2010) will find the turning points of the graph $f(x) = x^3 + 2x^2 + 1$ by setting $f'(x) = 0$ and solve for x using the quadratic formula, but they would not understand that the reason we set $f'(x) = 0$ is because the gradient is zero at the turning point. Table 3 provides illustrative examples on instrumental and relational understanding as explained by Skemp (1976) and Dunnigan (2010).

Table 3: Examples of instrumental and relational understanding

INSTRUMENTAL	RELATIONAL
1. Find the area of a rectangle with length = 100 m and breadth = 1000 cm. Student's response: $Area = 100\ m \times 1000\ cm$ $= 100\ 000\ cm^2$ Comment: The student did not realise that the units should be the same.	1. Find the area of a rectangle with length = 100 m and breadth=1000 cm Student's response: $1\ m = 100\ cm$ $\therefore 1000\ cm = 10m$ $\therefore Area = 100\ m \times 10\ m$ $= 100\ ^2$ Comment: The student converted the units first.
2. Solve for x: $x^3 - 2x^2 + 1 = 0$ Student's response: $x = \frac{-(-2)\pm\sqrt{(-2)^2-4(1)(1)}}{2(1)}$ Here students used the quadratic formula to solve a cubic equation.	2. Solve for x: $x^3 - 2x^2 - 3x = 0$ Student's response: $(x-1)(x^2-x-1) = 0$ $x = 1$ or $x = \frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)}$ $\therefore x = 1$ or $x = \frac{1+\sqrt{5}}{2}$ or $x = x = \frac{1-\sqrt{5}}{2}$ Here the students used the quadratic formula correctly.
3. When is $f'(x)g(x) > 0$? Student's response: $f'(x)g(x) > 0$ if $f'(x)$ is increasing and $g(x)$ is increasing. Comment: This is incorrect, since both the graphs are increasing in the interval $(45^\circ, 90^\circ)$ but $f'(x)g(x) < 0$.	3. When is $f'(x)g(x) > 0$? Student's response: $f'(x)g(x) > 0$ if: $f'(x) > 0$ and $g(x) > 0$ or $f'(x) < 0$ and $g(x) < 0$. Comment: This is correct, since the product of two negative quantities or two positive quantities is positive.

5. Methodology

Ethical clearance was obtained from the Research Office of the University of KwaZulu-Natal for the project, Online diagnostics for undergraduate mathematics (Protocol number: HSS/1058/01CA). The work for that project commenced in 2014 and is ongoing. Online quizzes were created to help first-year university mathematics students gauge their level of understanding of functions. The quizzes contributed to the calculation of the formative assessment component for the module on *Introduction to Calculus*. The quiz items were matching-type questions on functions (see Snapshots 1 and 3 in the next section). The design of the quiz items that students were exposed to was based on the suggestion of Orhun (2001) that trigonometric functions and their related concepts should be taught by using their graphs.

That was the rationale for the design of all quiz items on functions, that students were exposed to in Grade 12 during their schooling (see Snapshots 1 and 2 in the next section), and also to extend their understanding on the building on new functions (see Snapshot 3). Each question focused on the characteristics of the graphical representation of the relevant function. Once students determined all the characteristics for a given function, they could submit their answer and the system automatically gave a score. In two of the quiz items reported on in this study the graphs were provided and the students had to use those graphs to determine the relevant characteristics of the relevant functions. For the third question they had to use their knowledge of a standard trigonometric function to build a new trigonometric function, based on the reciprocal concept. For some sub-questions there was a need for algebraic working details before the student could answer; for example, see Snapshot 1. The participants for this study were first-year science students who enrolled for the module *Introduction to Calculus (Math130)*. This is a core module core offered by the School of Mathematics, Statistics and Computer Science to students enrolled for the Bachelor of Science degrees, offered by the College of Agriculture, Engineering and Science (CAES) at UKZN. The prerequisite requirements to enrol for the module are a Higher Grade D or Standard Grade A for Matric Mathematics, or NSC Level 5 Maths, or 60% for MATH199 (a foundational module), (University of KwaZulu-Natal, 2021a). Thus, in general, the module attracts the stronger cohorts of students. The three quiz items were administered online to students who were enrolled for the *Math130* module during the first three weeks, at the beginning of the semester. During that period, students were engaged with the first-year course material which included a brief revision of functions that they were exposed to during their schooling years. The relevant lecturers indicated to the researcher that after that brief revision three-week period, students were asked to take diagnostic quizzes that included the three quiz items. For the question on polynomial functions (see Snapshot 1) there were 446 attempts and for the questions on the trigonometric functions (see Snapshots 2 and 3) there were 557 attempts. The first attempts of those respective students were considered in the data analysis.

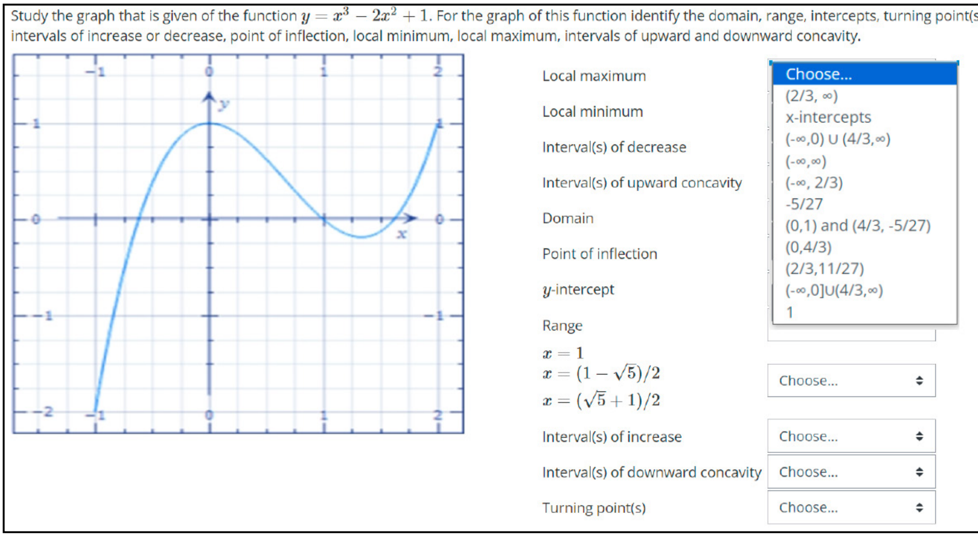
To find out how students performed in each sub-question, their responses for each sub-question were looked at and whether they chose the wrong or right option was noted. For each question the number of correct responses for each sub-question was noted. What was focused on in each sub-question was categorised, for example, domain, range, interval of increase/decrease and local extrema. For each category the percentages of correct responses for the relevant items were obtained from the Moodle platform that was used to administer the quizzes to students (University of KwaZulu-Natal, 2021b). Then the statistics arrived at were used in the context of the three quiz items, the literature review and conceptual framework to document the findings, analysis and discussion.

6. Findings, analysis and discussion

These are presented under the following subheadings: third-degree polynomial functions; trigonometric functions. Under each of these, the relevant quiz item is first given; to provide context for the findings, analysis, and discussion. In this section information indicated in the Snapshots and Tables was obtained from the website for the In-course Diagnostics for Calculus (University of KwaZulu-Natal, 2021b).

6.1 *Third-degree polynomial functions*

Snapshot 1 gives some of the questions based on a 3rd-degree polynomial function. Observe that this focuses on typical properties of a 3rd-degree polynomial function that a Grade 12 pupil is expected to be able to answer on; some basic properties while others are calculus related. For example, given the defining equation of the 3rd-degree polynomial function $y = x^3 - 2x^2 + 1$ the *x*-intercpts, *y*-intercept, domain and range which are basic properties; while finding the turning point(s), the intervals of increase or decrease and the nature of the local extrema are calculus-related concepts. The question was designed to address the shortcomings of Grade 12 pupils as detected in the literature review for the section on cubic functions; indicated in the diagnostic reports (Department of Education, 2014; 2019; 2020).



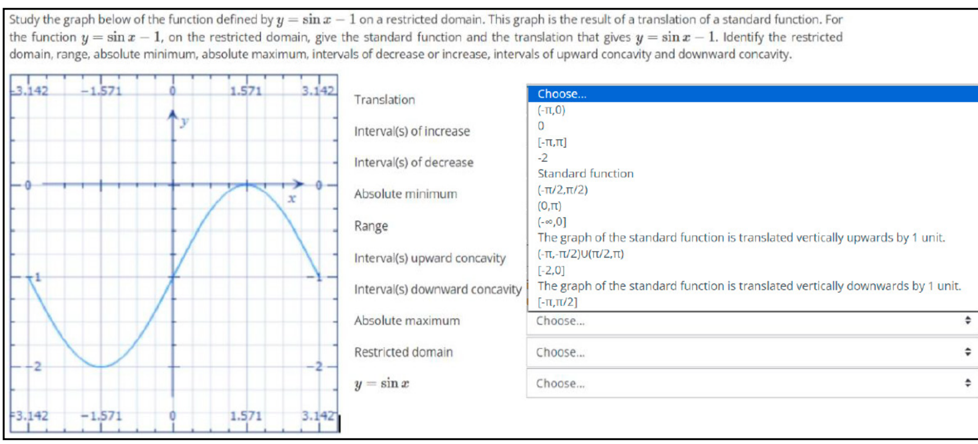
Snapshot 1: The online question on a third-degree polynomial function

Table 4: Percentage responses of students according to the characteristics of the third-degree polynomial function in Snapshot 1 (n = 446)

Characteristic	Correct response	Incorrect response	No response
Local maximum	89.46	10.54	0
Local minimum	90.58	8.97	0.45
Interval(s) of decrease	87	12.55	0.45
Interval(s) of upward concavity	88.79	11.21	0
Domain	96.86	2.92	0.22
Point of inflection	95.29	4.49	0.22
y-intercept	97.31	2.69	0
Range	95.29	4.71	0
x-intercepts	93.05	6.73	0.22
Interval(s) of increase	80.49	19.51	0
Interval(s) of downward concavity	87.89	12.11	0
Turning point(s)	97.98	2.02	0

Table 4 gives a summary of student response data on deductions from the graph of a third-degree polynomial function when given the graphical representation and the defining equation of the function, in the context of questions indicated in Snapshot 1. Observe that at least 90% of those candidates were able to answer correctly for the function under discussion on the following characteristics: the domain; the *x*- and *y*- intercepts; the range; the turning points. Further observe that for the following calculus related concepts correct responses were less than 90%: intervals of increase or decrease; point of inflection; intervals of concavity; local extrema. For each of these concepts note that there were a relatively small number of the candidates who did not give a response. In particular, about 13% to 20% of the candidates had difficulty with intervals of increase or decrease and concavity. This finding is in keeping with the diagnostic reports (DoBE, 2019; 2020). Further, it was encouraging that about 90% and above of the candidates were able to correctly answer the questions on turning points and local extrema. However, although about 98% of the candidates were able to find the turning points about 90% were able to correctly classify those turning points, implying that about 10% of the candidates lacked relational understanding in the context of local extrema. The implication from these observations is that instruction to incoming students at first-year level, should, when covering the calculus aspects focus on: (1) intervals of increase or decrease and concavity, both from graphical and algebraic perspectives, and (2) determining the nature of local extrema.

6.2 *Trigonometric functions*



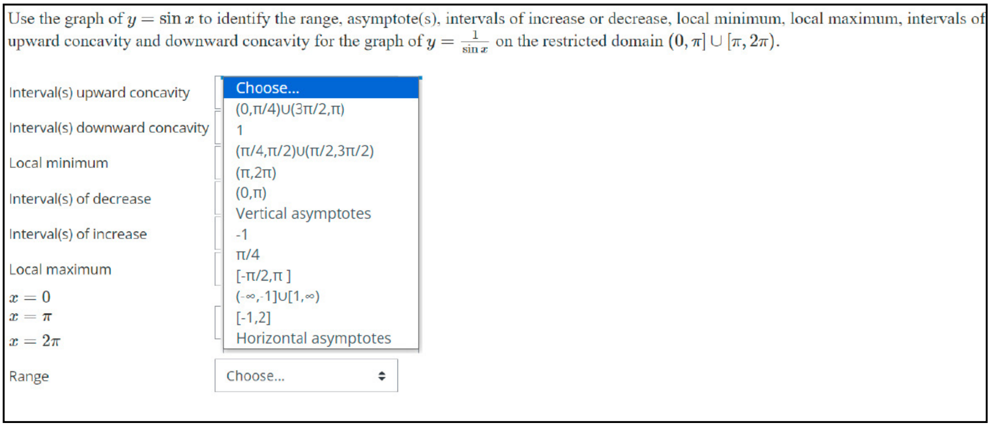
Snapshot 2: The online questions based on a trigonometric function

Table 5: Percentage responses of students to characteristics of the trigonometric function (n = 557)

Characteristic	Correct response	Incorrect response	No response
$y = \sin x$	97.13	2.51	0.36
Translation	93	6.64	0.36
Restricted domain	91.74	7.94	0.32
Range	90.66	8.8	0.54
Interval of decrease	89.59	9.69	0.72

Characteristic	Correct response	Incorrect response	No response
interval of increase	87.43	12.03	0.54
interval upward concavity	83.84	15.44	0.72
interval downward concavity	85.46	14	0.54
absolute minimum	92.46	6.46	1.08
absolute maximum	91.92	7.36	0.72

Table 5 summarises data on student responses to questions indicated in Snapshot 2. Although about 97% of the candidates were able to recognise that the standard function was represented by *y = sin x*, 93% of the candidates were able to correctly conclude that the standard function was translated by 1 unit downwards to arrive at the given graphical representation. Observe that at least 90% of the candidates were able to, for the function defined by *y = sin x – 1*, use the given graphical representation to find: the range and restricted domain; the absolute minima or absolute maxima. Note that the finding of this study on incoming university students' understanding of the domain concept differs significantly from that of Orhun (2001) indicated in the literature review section. Table 5 implies that about 10% to 17% of the candidates for this study had difficulties with using the given graphical representation of the function to find the intervals of increase or decrease and the intervals of concavity. This finding supports those indicated in the Diagnostic Reports for the Senior Certificate Examinations for Mathematics (DoBE, 2019; 2020). Table 5 also implies that for the types of questions indicated in Snapshot 2: (1) a very small percentage of the candidates, at most around 1%, gave no responses, and (2) about 17% of the candidates lacked relational understanding in the context of using the given graphical representation to find the intervals of increase/decrease and intervals of concavity correctly.



Snapshot 3: An online question on building new trigonometric functions

Table 6: Percentage responses of students to questions in the context of Snapshot 3 (n = 557)

Characteristic	Correct response	Incorrect response	No response
range	85.64	12.74	1.62
vertical asymptotes	93.9	5.02	1.08
interval increases	76.48	21.9	1.62
interval decreases	77.38	21.18	1.44
upward concavity	74.33	24.23	1.44
downward concavity	74.87	23.87	1.26
local minimum	74.51	24.23	1.26
local maximum	73.61	24.77	1.62

Table 6 summarises data on student responses to questions in Snapshot 3, which focused on students' ability to build new trigonometric functions. The quiz item in Snapshot 3 is based on the concept of a reciprocal of a function, where the denominator is not a constant but a function. It was designed to assess the students' relational understanding (Skemp, 1976) of functions. In the context of APOS (Dubinsky, 2010), it therefore focused on the higher mental structures, process level and higher. It is therefore understandable from Table 6, that relative to Tables 4 and 5, the percentages for characteristics focused on, are higher for incorrect and no responses. That could be connected to those students' ability to use a basic trigonometric function to build new trigonometric functions. It therefore supports the advice given in the diagnostic reports (DoBE, 2019; 2020) in the context of students lacking the understanding on how transformations or changes to an equation influences the graph. The implication is that there is a need for illustrations on how to develop a good understanding of how the graph changes when the equation changes and vice versa. Also note that in the current study about 85% of the students were able to deduce and select the range, and about 93% of the students were able to deduce and select the correct vertical asymptotes. Also observe that about three-quarters of the participants were able to deduce and correctly select for the new built function: (1) the intervals of increase or decrease, (2) the intervals of upward or downward concavity, and (3) the nature of local extrema. For these three concepts in the context of the conceptual framework the finding is that, for the building of new functions from familiar functions (of the type indicated in Snapshot 3), about 25% of the students lacked relational understanding (Skemp, 1976, Dunnigan, 2010) or mental structures at the higher levels of APOS (Dubinsky, 2010). This once again implies that there is a need for instruction on calculus to focus on those three concepts.

7. Conclusions

The use of functions and their related graphs to test students' knowledge of concepts related to functions that they encountered during their schooling years provided important insights on their level of understanding. Most of the participants were successful when answering questions that were general, non-calculus related questions, based on the characteristics of functions encountered during their schooling years. For all three quiz items about 10% to 25% of the participants had difficulties with the calculus related concepts: (1) the intervals of increase or decrease, (2) the intervals of upward or downward concavity, and (3) the nature of local extrema. For the first item on the third-degree polynomial where the graphical representation was given, the finding that about 13% to 20% of the candidates had difficulty with intervals of increase or decrease and concavity is in keeping with the diagnostic reports

(DoBE, 2019; 2020). The second item was based on the graphical representation of a translated basic trigonometric function. It was found about 97% of the candidates were able to recognise that the standard function was represented by $y = \sin x$. Further about 93% of the candidates were able to correctly conclude that the standard function was translated by 1 unit downwards to arrive at the given graphical representation. Those findings of this study differ from the diagnostic report (DoBE, 2014) which stated that there was a failure to detect the vertical translation that was applied to the basic trigonometric function. Furthermore, those findings indicate that participants of this study were able to recognise the algebraic meaning of vertical translations. This differs from the finding of Anabousy *et al.* (2014), which explains that their participants had difficulty giving algebraic meaning to certain transformations in the context of cubic functions.

In this study it was found that for each of the quiz items the difficulties increased for intervals of increase or decrease, intervals of concavity and the nature of local extrema. The understanding of these three concepts requires relational understanding (Skemp, 1976) or mental structures at the higher levels of APOS (Dubinsky, 2010; Maharaj 2013). The implication is that, for incoming university students, there should be a greater focus on these three calculus related concepts when the characteristics of functions are covered. Since about a quarter of the participants lacked relational understanding, there should be a concerted focus on developing relational understanding and the higher levels of APOS mental constructions for these three calculus concepts.

8. Recommendations

Since most students were able to find the turning point(s), while some of them had difficulty in classifying it/them as a local minima or local maxima, it is recommended that lecturers should plan for a differentiated instruction strategy in the contexts of lecturing/tutoring. Furthermore, it is recommended that instruction should clearly explain what a standard trigonometric function is and how it looks for a given defining equation. In particular there should be a focus with illustrations on developing a good understanding of how the graph changes when the equation changes and vice versa (DoBE, 2019; 2020). Also, it is recommended that more time should be focused on the building of new functions from functions, in the context of translations and the reciprocal of standard functions. It is also recommended that there should be a greater focus on why certain techniques work. Examples of these are: (1) Why do we set $f'(x) = 0$ when finding turning points of a function? (2) Why does one need to analyse the sign of $f'(x)$ in the neighbourhood of a critical value when determining the nature of local extrema?

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