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# Deepening pre-service secondary teachers' mathematical content knowledge through engaging with peers' mathematical contributions

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## Abstract

*Depth and rigour in mathematical knowledge for pre-service secondary school mathematics teachers may be found in particular experiences of learning and doing school-level mathematics. Drawing on two cases from a course on financial mathematics for secondary school teachers, I illustrate opportunities for exploring compound and exponential growth. I show that when some students unexpectedly produced a quadratic model for an exponential relationship, opportunities are opened up to study the usefulness of the quadratic function as a model of the given situation. Further opportunities are also opened up to explore the relationship between the quadratic and exponential functions, which in turn require students to draw on advanced mathematics. I also argue that the use of the compound growth formula in a situation where it is not typically used provides opportunity to deepen knowledge of the essential features of the formula. Finally, I reflect on how suitable opportunities for engaging with peers' mathematical contributions might be included in a pre-service programme for secondary school mathematics teachers.*

**Keywords:** maths-for-teaching, teacher knowledge, pre-service teacher education, compound growth, depth

## 1. Introduction

What do the notions of *depth* and *rigour* mean in preparing future secondary school mathematics teachers? Many might suggest it involves much attention to university-level mathematics. However, this has been challenged, partly on the grounds that advanced mathematics and the associated experiences of learning it do not prepare secondary school mathematics teachers well for the work they will do (e.g. Cooney & Wiegel, 2003; Stacey, 2008). In South Africa, there is the added complexity that many students who enter pre-service secondary school mathematics teacher education programmes are not well prepared for university mathematics<sup>1</sup>. One of the challenges facing secondary school mathematics teacher education, locally and internationally, is the appropriate selection of mathematical content and

providing appropriate mathematical experiences for students (Conference Board of the Mathematical Sciences [CBMS], 2001; Cooney & Wiegel, 2003; Stacey, 2008). Mathematical selections need to take into account the breadth of the South African school mathematics curriculum and thus include algebra, functions, calculus, Euclidean geometry, analytical geometry, trigonometry, financial mathematics, statistics and probability. This selection is broader than what is typically offered in undergraduate mathematics programmes. However, it does not include typical topics of undergraduate mathematics such as linear algebra, multivariable calculus, abstract algebra, group theory and real/complex analysis. The challenge of selection is not merely one of selecting topics. Former president of the International Mathematical Union, László Lovász, has suggested that mathematics education needs to consider seriously the recent changes in mathematical activity amongst mathematicians and the manner in which these might influence the dominance of classical approaches to mathematics in school and university curricula (Lovasz, 2008). Notable amongst these changes is the increasing use of computing technology and the resulting growth in applications of mathematics in new areas such as biological sciences and economics. He also argues that more attention needs to be paid to teaching students to communicate mathematics with their peers and with non-mathematical audiences, a sentiment strongly supported by Fields medallist, William Thurston (Thurston, 1994).

The practices of scholarly mathematics, school mathematics and mathematics teacher education are different. Consequently, notions of depth and rigour related to the mathematics of these different practices must be different. In this article, I argue that depth and rigour in mathematical knowledge for pre-service secondary mathematics teachers may be found in particular experiences of learning and doing school-level mathematics. I am not arguing that the study of advanced mathematics<sup>2</sup> is unnecessary but I am arguing, as do others (e.g. Stanley & Sundström, 2007), that school mathematics provides productive opportunities for deepening teachers' knowledge of mathematics.

I draw on data from a larger study where I designed and taught a semester-long course in financial mathematics to a group of pre-service secondary school mathematics teachers. I focus here on two instances near the beginning of the course and show that both have the potential to give student teachers experiences of learning mathematics to deepen their mathematical knowledge and develop an appreciation of the disciplines of mathematics and applied mathematics.

## **2. The mathematical content of secondary school mathematics teacher education**

Across the world, there is a variety of models for the mathematical preparation of secondary school mathematics teachers (Tatto, Lerman & Novotná, 2009). Despite a wide variety in the structure and content of programmes, there is some agreement on the goals of programmes. The following list has been compiled from a variety of sources (e.g. CBMS, 2001; Cooney & Wiegel, 2003; Stacey, 2008; Watson, 2008):

- students should gain knowledge *of* mathematics beyond the secondary school curriculum;
- students should gain knowledge *about* mathematics, including its history, philosophy, recent developments and populist ideas that stimulate public engagement in mathematics;

- students should develop a deep understanding of the core of school mathematics including the important procedures, algorithms and related skills;
- university mathematics should illuminate high school mathematics by providing an advanced perspective on it;
- students should have experiences of doing mathematics beyond the formalism of typical undergraduate courses, to include investigations, problem solving and mathematical modelling; and
- students should develop the skills to continue their own learning of mathematics and thus engage in mathematical inquiry as lifelong learners.

This is an ambitious set of expectations for any programme and even more so in the South African context where many aspiring secondary school mathematics teachers are underprepared to cope with university mathematics.

### 3. Secondary school mathematics teachers' content knowledge for teaching

When Shulman (1986) proposed the distinction between subject matter knowledge (SMK) and pedagogical content knowledge (PCK), he emphasised the importance that teachers' knowledge should include knowing *that* as well as knowing *why*. In mathematics, this knowledge extends beyond definitions, theorems and algorithms to understanding the structure of mathematics as a discipline, the relative importance of particular mathematical ideas within the discipline and the principles of mathematical inquiry – how new ideas are added to the body of knowledge and erroneous ideas are rejected.

In this article, I focus particularly on teachers' *mathematical* content knowledge for teaching. Although the notion of PCK has gained wide acceptance in the mathematics education research literature and in teacher education more broadly (Ball, Thames & Phelps, 2008), clear analytical distinctions between SMK and PCK are not easy to make (Huillet, 2007; Petrou & Goulding, 2011). I therefore adopt the notion of maths-for-teaching (MfT) (Adler, 2005; Adler & Davis, 2006) which I consider as an amalgam of mathematical and pedagogical knowledge. I drew on several existing frameworks of teachers' mathematical knowledge (e.g. Ball *et al.*, 2008; Even, 1990; Ferrini-Mundy *et al.*, 2006; Huillet, 2007) to propose a framework of nine aspects which can be grouped into three clusters: 1) aspects that are mainly mathematical; 2) aspects that are mainly pedagogical; and 3) a cluster related to financial mathematics, which spans knowledge of financial concepts and conventions, socio-economic issues and financial literacy. Since this article focuses primarily on mathematical issues, I elaborate only on the four sub-components of the mathematical cluster.

- Knowledge of the *essential features* of a concept (Even, 1990, 1993) which includes identifying the concept and its representations (*what* it is) and executing associated techniques (*how* to work with it).
- Knowledge of the *relationship of the concept to other mathematics* that is primarily concerned with making connections between different mathematical concepts and topics.
- Knowledge of *mathematical practices* relates to Shulman's (1986) initial conception of teachers' content knowledge regarding how knowledge is established and grows in a particular discipline. Watson (2008) provides an extensive list of mathematical practices typical of academic mathematics, including empirical exploration, conjecturing, logical

deduction, verifying, formalising and reflecting on answers which may lead to further inquiry, comparing solutions for accuracy, validity and efficiency; reworking to identify technical and logical errors and presenting problems and their solutions to others. While scholarly mathematics and school mathematics are different practices it is experiences of such practices that lead to deepening teachers' mathematical knowledge and an appreciation for mathematical rigour.

- Knowledge of *mathematical modelling and applications* – since the study focused on financial mathematics, modelling and applications was separated from other mathematical practices to avoid drowning all descriptions of mathematical practices in modelling.

#### 4. Deepening, broadening and connecting mathematical knowledge

Ma (1999) proposed that a profound understanding of mathematics consists of three inter-connected components: depth, breadth and thoroughness, all of which are related to the structure of mathematics. Depth of understanding concerns the ability to connect a concept or a topic “with more conceptually powerful ideas of the subject” (*ibid*: 121) where the power of a mathematical idea is related to its proximity to the structure of the discipline. A mathematical idea that is closer to the structure of the discipline underpins more topics and hence has more “mathematical influence” (my term) and thus mathematical power. For example, a teacher with a deep understanding of number and number operations will link the notion of subtraction-with-regrouping to the core idea that addition and subtraction are inverse operations.

By contrast, breadth of understanding involves connecting a concept or topic with topics and/or concepts of similar or less conceptual power, for example connecting subtraction-with-regrouping to addition-with-carrying. The notion of thoroughness concerns the ability to make connections. Ma (1999: 121) argues that “it is this thoroughness which ‘glues’ knowledge of mathematics into a coherent whole”.

Drawing on these notions of depth, breadth and connectedness, I have developed a hierarchy of concepts relating to compound interest and annuities (Pournara, 2013, 2014). Following Ma (1999), I have argued that percentage is a powerful idea and that linear growth, exponential growth and progressions (arithmetic and geometric) can be placed at consecutively higher levels within the hierarchy<sup>3</sup>. In addition, I have argued that the notion of *growth factor* is a key component of compound interest and lies between percentage and linear growth. Briefly, growth factor represents the multiplicative entity by which an amount increases or decreases. It is derived from percentage increase/decrease and is easily illustrated using the example of value-added tax (VAT). If an item costs R80 excluding VAT, then its price including VAT can be calculated as:  $80+80\times 0.14 = R91.20$  or  $80\times 1.14 = 91.20$ . The entity 1.14 is the growth factor. In a more general case of an amount P increasing by a rate r%, the growth factor would be  $(1+r\%)$ .

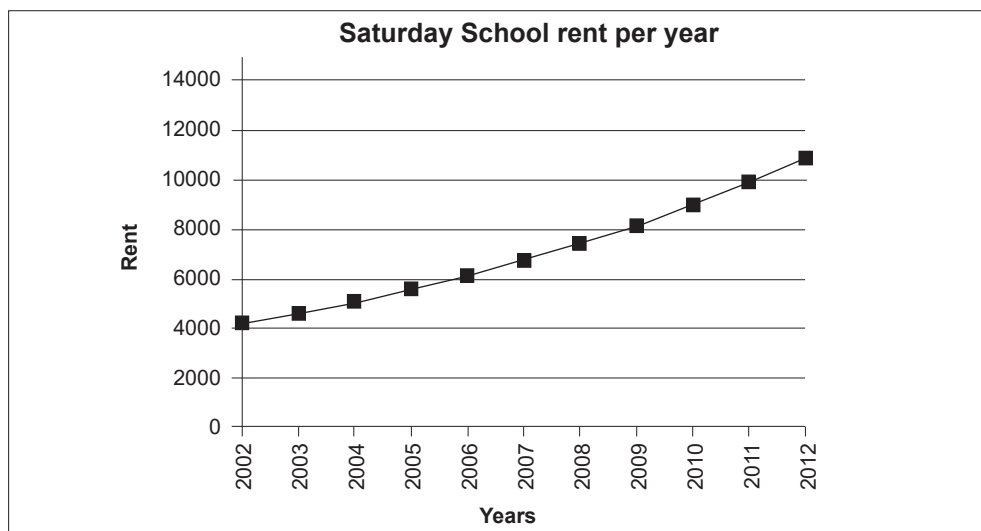
In the next section, I provide two cases through which I illustrate opportunities for pre-service secondary school mathematics teachers to deepen their MfT and to experience the rigour of mathematics and applied mathematics. I am not claiming that all this was achieved in the course. I am arguing that the potential exists for the students to learn based on my analysis of the data and reflections as the teacher-researcher.

## 5. Opportunities for pre-service secondary school mathematics teachers to deepen their MFT

### 5.1 Case 1: Extending a task by engaging with students' productions

In the course, students were allocated to tutorial groups and worked together on a task each week, having to submit a combined report on their work at the end of the session. In the first group tutorial, students were given the following scenario, referred to as the *Saturday School Rent Problem*, and the accompanying graphic which represents an attempt by three fictitious students to answer the question.

*A Saturday School pays rent for the building they use each week. In 2002, they were paying R4200 per year. At the end of the year, the owners of the building met with the school management and agreed to a 10% increase in rent each year from 2003 onwards. How much will the Saturday School pay each year from 2002 to 2010?*



**Figure 1:** Graph for hypothetical learners' response to Saturday School Rent Problem

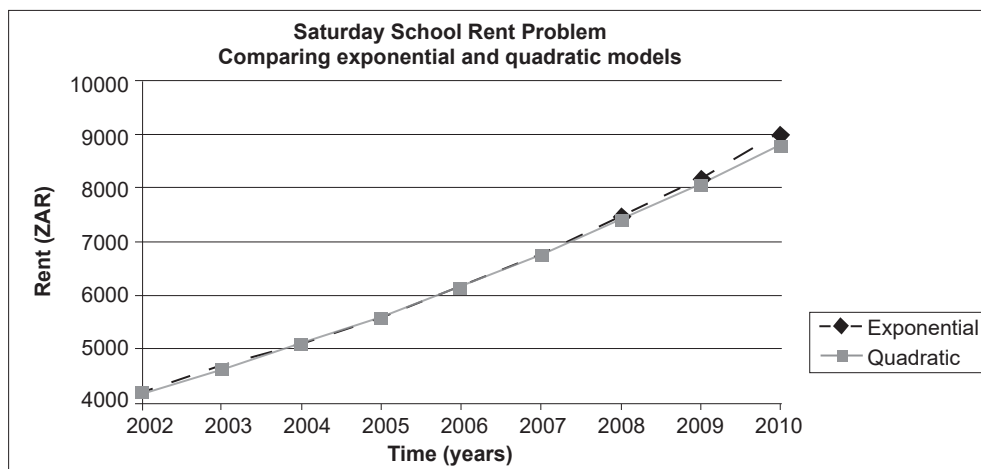
Students were required to answer the question and find a possible equation for the given graph. I had expected them to recognise that the situation involved exponential growth and hence to produce formulae of the form  $f(x)=a.b^x$  such as  $f(x)=4200(1.1)^x$  where  $x$  is the number of years after 2002 and the growth factor is 1.1. Most groups produced this formula or one similar in structure. However, one group produced the following formula  $h(x)=21.7x^2+398.3x+4200$ . This came as a surprise since the quadratic structure of the formula does not model an exponential scenario. However, there were no computational errors in the students' report and so I was intrigued to investigate it further.

Table 1 and the accompanying graph (fig. 2) show that the quadratic function is a very good model for a certain time interval. For example, the percentage error (i.e. the difference between the exponential and quadratic values, expressed as a percentage) is less than 1% each year from 2002 to 2008, and for the next 2 years, it is less than 3%. The graphical representation shows that the two lines are almost on top of each other from 2002 to 2007, and the gap between them is

still very small in year 10. However, as can be seen in the table, the gap widens substantially, and by 2020, the percentage error is more than 20%. Thus, we can see that the quadratic function is a good model for predicting the rent over a certain period, but then becomes a very poor model.

**Table 1:** Comparison of numeric values for exponential and quadratic models, and percentage error

Year	Exponential model	Quadratic model	Difference	% error
2002	4200.00	4200.00	0.00	0.00
2003	4620.00	4620.00	0.00	0.00
2004	5082.00	5083.40	-1.40	-0.03
2005	5590.20	5590.20	0.00	0.00
2006	6149.22	6140.40	8.82	0.14
2007	6764.14	6734.00	30.14	0.45
2008	7440.56	7371.00	69.56	0.93
2009	8184.61	8051.40	133.21	1.63
2010	9003.07	8775.20	227.87	2.53
2011	9903.38	9542.40	360.98	3.65
2012	10893.72	10353.00	540.72	4.96
2013	11983.09	11207.00	776.09	6.48
2014	13181.40	12104.40	1077.00	8.17
2015	14499.54	13045.20	1454.34	10.03
2016	15949.49	14029.40	1920.09	12.04
2017	17544.44	15057.00	2487.44	14.18
2018	19298.89	16128.00	3170.89	16.43
2019	21228.78	17242.40	3986.38	18.78
2020	23351.65	18400.20	4951.45	21.20



**Figure 2:** Graphical comparison of exponential and quadratic models

### 5.1.1 Reflecting on the opportunities for deepening knowledge

This task and the students' unexpected response provides many opportunities for further mathematical work, some of which involves mathematical modelling and other work that requires further investigation of the quadratic function per se. Inevitably, students would need to begin by investigating how the values 21.7, 398.3 and 4200 were obtained. The mathematics required to do this is within the scope of the secondary school curriculum although these values are "messy" and are not typical of the integer-values frequently encountered in school textbooks.

With regard to modelling, the most obvious learning opportunity is that students are challenged to consider alternatives to the expected exponential model. Thereafter they would need to investigate the domain (in this case the timeframe) over which the model is a useful one thus requiring them to engage with the notion of error in mathematical modelling, a fundamental aspect of rigour in applied mathematics. Thereafter students might investigate other quadratic functions (with different coefficients and constants) and compare these as models of the situation. An important modelling consideration here would be the choice of points to establish the equation and hence an opportunity to contrast interpolation (estimating values between the chosen points) and extrapolation (estimating values outside the interval of the chosen points).

Apart from modelling, there are several opportunities to investigate the exponential and quadratic functions per se. For example, are the functions coincident over a particular domain? In terms of the graphs, this would mean that the graphs lie on top of each other. While it can be shown that the functions are not coincident and have exactly three points of intersection, the mathematics to do this is beyond the scope of school mathematics. This stimulates opportunity to introduce the necessary numerical methods (e.g. Bisection method or Newton-Raphson method) to solve the problem. This in turn introduces students to some aspects of advanced mathematics in a way that connects directly to mathematical problems emerging from deeper exploration of school mathematics. Students can also draw on computer applications such as spreadsheets and dynamic graphing software to investigate the points of intersection<sup>4</sup>.

In the above discussion, I have illustrated the potential that lies in working with an unexpected response from the student teachers. I have shown that while this investigation begins in school-level mathematics, it easily extends beyond the domain of school mathematics and ultimately prompts the need for tools of advanced mathematics to solve a problem that arises spontaneously in school mathematics. By engaging with such a task, students have an opportunity to develop MfT with particular emphasis on mathematical modelling, applications and mathematical practices more broadly.

## 5.2 Case 2: Engaging with a peer's work-in-progress

The second case provides an example of a student's self-initiated attempt to make new connections between various aspects of exponential growth and the difficulties that emerge as he seeks the help of his peers in resolving a problem that had arisen. It also exemplifies the potential to engage in further mathematical inquiry that is directly related to school mathematics.

In a previous session, students had worked on the *Wage Doubling Problem* where a worker's wage was doubled each working day, starting with 1c on the first day, 2c on the second day etc. Students produced exponential equations to model this scenario. They were

then asked to represent the problem in terms of percentage change. There was consensus that the percentage increase was 100% per day. However, similar to the previous case, one group suggested the percentage increase was 200% per day. They presented a compelling argument and many students were not immediately able to identify the logical error in their reasoning. The following day, having grappled with the explanation in their own time, students could identify the logical error and recognised that 200% represented the ratio of the amount earned on the current day to the amount earned the previous day and not to the *increase* from the previous day. For example, if the worker earned 8c yesterday, he would earn 16c today. The increase is 8c and the percentage increase is 100% but the ratio 16/8 is 200% when expressed as a percentage.

Following from this, Sakhile initiated his own investigation into the relationship between the doubling formula  $y=2^{n-1}$ , and the compound interest/growth formula,  $A=P(1+i)^n$  while drawing on what I shall call the *200% formula*, i.e.  $200\%=x+100\%x$ . In seeking to make explicit links between the different formulae, he had established that the basis of the doubling formula and the compound growth formula could be shown to be the same. However, the exponent of the doubling formula was  $n-1$  while the exponent for the compound growth formula was  $n$ . He was unable to account for the difference in the exponents and was seeking help from the class to resolve the impasse.

I invited Sakhile to share his work with the class. His board work is shown below (fig. 3a and fig. 3b). The layout of his board work did not easily support his explanation and so I have introduced a different layout (fig. 3c) in an attempt to show the links and his reasoning more clearly. The lines in the new layout are labelled A-E. I refer to lines using the notation [letter], e.g. [C] is line C.

The image shows a chalkboard with the following handwritten equations:

$$200\% = x + 100\%x$$

$$= x(1 + 100\%)$$

**Figure 3a:** Links between 100% and 200%

The image shows a chalkboard with the following handwritten mathematical expressions:

$$2^{n-1}$$

$$P(1+i)^n$$

$$(1+100\%)^{n-1}$$

**Figure 3b:** Links between exponential form and compound interest formula



200% formula		Doubling formula		Compound interest formula	
A	$200\% = x + 100\% x$	C	$2^{n-1}$	D	$P(1+i)^n$
B	$= x(1+100\%)$			E	$(1+100\%)^{n-1}$

**Figure 3c:** Reformatted version of Sakhile's board work

Sakhile began by showing that 200% was obtained by adding the initial salary,  $x$  and a hundred per cent of the salary [A] (fig. 3a). He then factorised the expression, producing [B], which he noted had similarities with the form of the compound interest formula. He then shifted to the exponential form, recalling that the formula for daily wage on day  $n$  was given by  $2^{n-1}$  [C] and then wrote the "right hand side" of the compound interest formula, i.e.  $P(1+i)^n$  as shown in [D]. Having written down all three forms, his intention was to describe and justify the links he had established, and to seek help with the exponent.

In his next move, he appeared to be treating  $2^{n-1}$  and  $P(1+i)^n$  as partial templates for [E]. He first substituted  $P=1$  because the starting wage was 1c. Then he focused on the need for the bracket in [D] to have a value of two. This would occur if he substituted 100% for  $i$ . He then appeared to treat [C] as the template and copied the exponent  $(n-1)$  to produce the expression  $(1+100\%)^{n-1}$  in [E]. He knew that the exponent in [E] did not match the exponent in [D] and this is where he sought the class's help.

### 5.2.1 Class discussion of case 2

Many students had not followed Sakhile's reasoning and were more concerned with *what* he had written rather than with his reasoning. There were two main concerns: the exponent in [E] and the lack of a referent for the percentages in [A], [B] and [E].

#### Concern about the exponent of $n-1$

Sizwe questioned why Sakhile was "subtracting one from the  $n$ " which indicated that he had not heard Sakhile's own concern about the exponent. Sizwe's concern was not surprising since [E] is not logically derived from [D] and when one looks at the vertical layout of the three lines in fig. 3b, one might expect the lines to represent some form of logical derivation. Sakhile then attempted to explain once more.

Sakhile:	Okay like I said, I tried to take this, sort of the formula that we had ( <i>pointing to <math>2^{n-1}</math></i> ), we got about the story, ja, the story about the salary, ja, relate it to the compound formula. So I'm, I'm shifting from this, sort of an exponential to, to the compound formula ( <i>pointing to <math>P(1+i)^n</math></i> that we know in textbooks. So I tried to change, so I know this ( <i>pointing to <math>2^{n-1}</math></i> ), since this ( <i>pointing to <math>(1+100\%)</math></i> ), it's here ( <i>pointing to the 2</i> ) and then what about this part ( <i>pointing to the exponent in <math>2^{n-1}</math></i> )? Where is it here from that formula ( <i>referring to the compound growth formula</i> )? Then I tried, okay it belongs ( <i>pointing to exponent in <math>(1+100\%)^{n-1}</math></i> )... that's how I, that's how I worked.
Craig:	And you said you're not sure why ...
Sakhile:	Ja, the negative one I didn't, like, why it's negative one, cos the compound is only $n$ ( <i>i.e. the exponent is <math>n</math></i> ).

Sakhile explained again how he was trying to make links between the doubling formula and the compound growth formula but students still struggled to make sense of his ideas and did not appear to be able to help him resolve the exponent issue.

*Lack of a referent for 100%*

Several students were concerned that Sakhile had not paid attention to a referent for the percentages. Mpho questioned whether we could add together 1 and 100% without there being a referent for 100%:

Mpho:	Okay, I don't know, maybe I'm, I'm lost. Can you actually say 'one plus hundred per cent' or do you have to say 'one plus one hundred per cent <b>of something</b> ' or do you just say 'one plus one hundred per cent, one plus two hundred per cent', can you actually understand?
Sakhile:	Okay, like, like, you see here ( <i>pointing to <math>x+100\%x</math></i> ). The way I see it, it's like $x$ plus hundred per cent of something, but when I try to factorise it, take out the like terms, it becomes like $x$ <b>into</b> one plus hundred per cent.
Mpho:	Please clarify, okay I think I understand, but I mean there ( <i>pointing to right section of board</i> ) where you say one plus hundred per cent.
Sakhile:	Here, I know, one hundred per cent. Uh, it's one, so I try to, that's how I try to take, fit, like, make this information like, link it to the two. Link it to the two ( <i>i.e. a base of 2</i> ). I know the one is there, but what is it that I must add, which is a percentage so that I can get the two?

Sakhile referred back to [B] where he had factorised the expression, producing a factor of  $1+100\%$ . He noted that there was a referent,  $x$ , which had been factorised as a common factor and had been substituted with a value of 1. So he was able to provide a logical justification for moving from  $x+100\%x$  to  $1+100\%$ .

Later Hailey drew Sakhile's attention back to  $200\%=x+100\%x$  and used an example to illustrate the problem with the lack of a referent for  $x$ .

Hailey:	But what, even about that top line? How is that equal to two hundred per cent?
Sakhile:	You mean this one?
Hailey:	Mmm.
Sakhile:	So ...
Hailey:	... Cos pretend, you were saying earlier that $x$ is eight, right?
Sakhile:	Ja?
Hailey:	So let's say $x$ is eight, and then we say a hundred per cent of eight is eight, eight plus eight is sixteen.
Sakhile:	Sixteen.
Hailey:	Is that equal to two hundred per cent?
Sakhile:	Yes, cos it will take from, for, for that day that's the total, I said the $x$ is the amount from the previous day.
Hailey:	That's <b>not</b> equal. It's equal to two hundred per cent <b>of <math>x</math></b> .

Hailey showed how Sakhile's equation  $200\%=x+100\%x$  would produce the "equation":  $200\%=16$ , where the left-hand side was not equal to the right-hand side. She then emphasised the need to include the referent on the left-hand side, i.e.  $200\% x$ . Sakhile could not see her problem at this point. It appears that he was using  $200\%$  as a *label* and was not intending to express some kind of equivalence. So he could equally have written: double salary =  $x+100\%x$ .

The responses to Sakhile reflect the difficulty of students working together on “maths in progress”. While he sought help to resolve the differences in the exponents, many were focused on his sloppy use of notation. It is not clear whether the students' concerns with his notation hindered them from engaging with his primary concern or whether they resorted to focus on notation because they were unable to provide help on the exponent issue.

### 5.2.2 Reflecting on opportunities for deepening knowledge

Sakhile was concerned with reconciling the two formulae and the manner in which they modelled the wage-doubling scenario. While I followed his reasoning and the disconnects he was grappling with, it was not immediately obvious to me how to resolve it. After the session, I sat down to investigate the problem. Table 2 shows my initial attempt to model the wage-doubling problem using the form  $2^n$  and using the compound growth formula  $A=P(1+i)^n$ . Whereas Sakhile had focused on the signifiers within the formulae, I chose to focus on modelling the situation.

**Table 2:** Modelling Wage Doubling Problem using exponential form and compound growth formula

End of day	Wage for day	Exponential form	Compound interest formula
1	1c	$2^0$	$P=1$
2	2c	$2^1$	$A=1(1+100\%)^1$
3	4c	$2^2$	$A=1(1+100\%)^2$
4	8c	$2^3$	$A=1(1+100\%)^3$
$n$		$2^{n-1}$	$A=1(1+100\%)^{n-1}$

In both cases I focused on the wage at the end of day  $n$ , showing that the exponent is always one less than the day number and thus both models *should have* the same exponent of  $n-1$ . However, this does not address the mismatch in the exponents in Sakhile's version. After further grappling I realised that in the usual context of compound growth, the present value,  $P$ , is the value at the *beginning* of the first period. Therefore, one period's interest has accumulated by the end of the first period. Consequently, the exponent will match the number of periods. However, in the wage-doubling scenario, the present value is the wage *at the end* of day 1, i.e. no increase has yet occurred. Consequently, when using the compound interest formula to model the doubling scenario, the exponent will be  $n-1$  because there is no compounding in the first period. One can also think about it as follows: the exponent,  $n$ , represents the number of compoundings. Since no compounding has taken place by the end of day 1, the exponent will always be one less than the day number.

It was only in working with the compound growth formula for a problem-type where it is *not* typically used that I became explicitly aware of the different ways in which different scenarios with exponential structure such as compounding of interest, wage doubling and bacteria growth work with different starting conditions and hence are modelled by slightly different formulae. This foregrounded the obvious meaning of present value, which I had taken for granted, that present value indicates the value at the *beginning* of the first period. This is so tacit when using the formula in its usual context that I was unable to draw on it immediately to resolve Sakhile's concern. Thus, through the use of the formula in an unusual context, an essential feature of compound growth that was invisible, now became visible

(Lave & Wenger, 1991). This suggests that using familiar mathematics in unfamiliar ways may be a productive strategy for deepening knowledge of school mathematics.

The above case provides an example of mathematical practices that are not generally made visible in undergraduate or school mathematics (Burton, 2004; Watson, 2008). We see an instance where a student asks his peers for assistance with his own mathematical work-in-progress. Thus, mathematics is shown to be a collaborative enterprise to solve a student-initiated inquiry and not merely the provision of answers to questions posed by the teacher or teacher educator.

The discussion of the episode would be incomplete without some attention to the difficulties students experienced in following Sakhile's argument and his inability to make his concern clearer for them. It could be argued that the students did not share his concern sufficiently to engage deeply with it, and therefore focused on relatively superficial aspects rather than engaging with the mathematical substance of his work. However, it could also be argued that the quality of his explanation and accompanying written text did not encourage them to engage. Irrespective of the reasons for students' lack of engagement, this example shows that it is not trivial to set up a situation where students engage substantively with each other's work-in-progress, in the ways proposed in the literature.

Nevertheless, the concerns around Sakhile's use of percentage notation led to a teachable moment (Havighurst, 1972) for me to discuss with students aspects of mathematical communication such as presentation of ideas and use of mathematical symbols. In order to provide a convincing argument, one needs to communicate it in such a way that the logic is clear and this involves precise use of notation and mathematical language. Since Sakhile was presenting work-in-progress, it was inevitably less rigorous and precise. However, without a minimum level of clarity and precision, his peers struggled to make sense of his problem and hence were unable to help him move forward.

## **6. Conclusion**

I have argued that working on school-level mathematics can provide opportunities for pre-service secondary school mathematics teachers to deepen their MfT and to appreciate the rigour of mathematical practices, both pure and applied. In both cases described above there was potential for attention to the essential features of exponential growth and to the kinds of mathematical practices proposed in the literature. Since students are familiar with the mathematics, they can work more flexibly with it. Therefore, the challenge is not about what the mathematics is but about how one uses it and then communicates about it. Typically, mathematics content courses are characterised by covering new content at a high pace and so students, apart from those who are very strong mathematically, have little time to consider how to use it – they are much more focused on mastering the content that will be assessed.

The opportunities described above are dependent on a pedagogy and a course design that is sufficiently flexible to incorporate students' mathematical explorations into the programme, to make them available to other students and then to engage with them in a public forum. Herein lies the challenge. Firstly, such practices require additional time, which in turn influences content coverage and pacing. Secondly, in which courses might one engage in such practices with pre-service teachers? If mathematics courses are offered in mathematics units, there is little chance of adopting such an approach given the large class sizes and vast amounts of

content to be covered. If, on the other hand, mathematics courses are offered specifically for teachers then such a practice becomes more possible. However, my experience has shown that even in such courses, the opportunities to deal with school-level mathematics in the ways described above may be limited. Frequently school mathematics is included in methodology courses not the mathematics content courses. However, it is unlikely that the kinds of mathematical work described above will take place in methodology courses because the focus there is on pedagogical issues. Consequently, one cannot imagine lecturers teaching numerical methods such as Newton-Raphson to find the points of intersection of exponential and quadratic functions in a methodology course.

In a climate of increasing pressure and accountability in teacher education and higher education more broadly, there is less and less opportunity to be flexible and responsive to students' work within a course. While it may be possible to design a limited number of such opportunities into a course, one cannot necessarily pre-specify the mathematical content since it needs to be responsive to students' contributions. Therefore, the content becomes mathematical practices rather than mathematical concepts, where the concepts are the means to engage in particular mathematical practices. This remains a challenge of mathematics teacher education, not only in pre-service programmes but in professional development too.

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## (Endnotes)

- 1 This is well known amongst those working in pre-service mathematics teacher education across the country. While I make this statement based on personal experience and from conversations with colleagues at other universities, I am not aware of any published research to support this claim specifically in relation to pre-service secondary school mathematics teachers.
- 2 Following Zazkis and Leikin (2010) I consider *advanced mathematics* as the mathematics that is learned in undergraduate mathematics courses (and beyond).
- 3 If one were to extend the hierarchy below percentage, it would be connected to multiplicative reasoning, which lies at the core of the structure of mathematics.
- 4 I have worked with students on such tasks in a subsequent course on financial mathematics.