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Lizzy's struggles with attaining fluency in multiplication tables

Sarah Bansilal

Many learners struggle to make the transition from addition and subtraction to multiplication and division, which hampers further progress in mathematics. In this article, I present a case study of one learner who struggled to attain fluency in multiplication by seven. The purpose of this study was to identify and explain how previous non-encapsulations related to addition and subtraction number bonds hampered her efforts in attaining fluency in the multiplication tables. Data for the study were generated from observations and interactions with the child, Lizzy, over a period of two years. The study employed a narrative analysis technique, while drawing upon the APOS (action, process, object, schema) framework together with the notion of conceptual embodiment. The findings suggest that Lizzy's struggles were due to a non-encapsulation of the various number bond strings, which did not allow her to see patterns in addition by seven. By using an alternative intervention, Lizzy was able to use this strategy as a conceptual embodiment leading her to greater fluency in the seven-times table.

Introduction

Although most learners spend much time in early years of numeracy on practising addition and multiplication, many struggle to reach the kind of fluency that could make their calculations less burdensome. Ernest (2006: 89) says fluency in addition is reached only after several years of practice, when a child knows all of the 100 one-digit addition facts and is able to retrieve any of them in a 'single rapid mental operation'. Achieving this fluency can be made less painful if children were able to compress some of the processes into 'derived facts' which is distinguished from 'a rote learned fact by virtue of its rich inner structure' (Gray & Tall, 1991: 3).

Gray and Tall (1991) discuss how progress from counting all to counting on can be hampered when compression to derived facts does not take place. Their discussion starts by considering a number, e.g. 'three', as representing the process of counting 'one, two, three' and can also be seen as the outcome of that process (Gray & Tall, 1991: 2–4). Usually, the counting-all conception is transformed or compressed into a conceptual entity when the child can see the number 3 as an object. If this transition

Sarah Bansilal,
University of KwaZulu-Natal
E-mail: Bansilals@ukzn.ac.za
Telephone: 031 260 3451

does not take place, the child's progression to the counting-on conception of addition will be hampered. Consider the sum of two numbers, e.g. 3+2. This can be seen as a process of counting all or counting on. A child whose conception of 3 is as a process will have available to her only the counting-all strategy because, with the counting-on strategy, one takes 3 as an object and then counts on from the 3. A child who is still working with a process conception of 3, can still reach an answer for 3+2 but, in so doing, she will use a more cumbersome procedure which involves counting-all the three, then continuing and counting-all the two. By the time she reaches the end of the process, she may have forgotten the beginning; thus, the sum of 3+2 will not be available to her as an object that could be used in further processes such as in subtraction or in sums such as 13+2, which depend on an understanding of place value as well. As the demand in complexity increases, the strategies used by children who have not moved beyond counting all, lead to 'intolerable difficulties and a high probability of failure' (Gray & Tall, 1991:4), while the child who has encapsulated the operations is able to move on to greater fluency.

In this study, I consider the case of one child who spent more than two years practising the various number bonds in early primary school. However, her understanding of these bonds remained at a process level and did not allow her to work at higher levels by using them as derived facts. Consequently, her methods of working out the eight-times and seven-times table were laborious and complex. I will describe how an alternative path was followed to develop the fluency she required in using these time table strings.

Theoretical framework

Simon, Tzur, Heinz and Kinzel (2004: 306) consider learning mathematics as 'a process of transforming one's ways of knowing (conceptions) and acting'. Dubinsky's (1991) APOS (action, process, object, schema) theory provides a more precise description of these transformations:

> *[A]n action conception is a transformation of a mathematical object by individuals according to an explicit algorithm which is conceived as externally driven. As individuals reflect on their actions, they can interiorize them into a process. Each step of a transformation may be described or reflected upon without actually performing it. An object conception is constructed when a person reflects on actions applied to a particular process and becomes aware of the process as a totality, or encapsulates it. A mathematical schema is considered as a collection of action, process and object conceptions, and other previously constructed schemas, which are synthesized to form mathematical structures utilized in problem situations (Trigueros & Martínez-Planell, 2010: 5).*

Within APOS theory, a person's understanding of a concept can be transformed from a process into a totality (object) upon which other transformations can act. When these objects are operated on in further processes, the cycles add layers in the system of mathematical concepts. If the previous layer is available to a learner as an encapsulated object, then building upon it becomes a little less burdensome.

De Lima and Tall (2008) propose (using Piaget's work) two different ways of operating on the world; the first is the process-object encapsulation development described by Dubinsky (1991) and the second is what they refer to as a conceptual embodied world arising from concrete or physical representations of a mathematics operation or concept. The authors distinguish between a conceptual embodiment and procedural embodiment. Many learners develop procedural embodiments arising from a concrete embodiment which may work for one situation but, in trying to extend it to other situations, they develop misconceptions. Conceptual embodiment, on the other hand, refers to a concrete embodiment that is productive in enabling a shift from the concrete to the symbolic which, in turn, enables a further process-object development in understanding. In their study, De Lima and Tall (2008) suggest that a robust concrete embodiment can develop into a conceptual embodiment, which will then facilitate further movement in understanding.

The study

The purpose of the study was to identify reasons why one learner struggled to develop fluency when working with the multiplication tables. The participant, Lizzy (a pseudonym), was between 9 and 10 years old at the time of the study. Data were generated from Lizzy's written and verbal responses to various questions, Lizzy's school workbooks and my own reflections (comprising field notes) as I worked with Lizzy trying to improve her fluency in the multiplication tables. The process followed in constructing a vignette of Lizzy is what Polkinghorne (1995) described as narrative analysis. Outcomes were identified and I went back to the data and identified 'thematic threads' relating to those outcomes (Polkinghorne, 1995: 12). These thematic threads relating to the conception of the multiplication table were then configured into the vignette.

The analytical framework comprising the genetic decomposition of the concepts is presented next.

A genetic decomposition of multiplication tables

An essential construct of APOS theory is a theoretical description called a genetic decomposition. This is used to 'characterise the linkages and representations within a concept' (Meel, 2003: 154). A genetic decomposition 'provides a possible path for a learner's concept formation; however it may not be representative of the path taken by all students' (Meel, 2003: 154; Dubinsky, 1991). In this framework, I present a possible genetic decomposition that can be used to characterise progression from single addition facts to multiplication. I refer to the progression as encapsulation paths, and the various elements of the genetic decomposition are called layers. However, it is important to note that this is a possible path for a learner's conception, and it will be shown that this path did not work particularly well for Lizzy's own conception.

Layer 1: Process-object understanding of single number bonds

A first step is understanding single number bonds or addition facts, e.g. the sum of 5 and 7. At a process level, learners would be able to work out each single element (say, 5+7=12) belonging to the number bond string of, say, 12. When learners can use other forms of the addition fact to perform manipulations upon it such as being able to answer the questions '5+ what = 12',or '12–5 = what?' they show evidence of being able to see the addition fact as a whole – suggesting that they have an object understanding of the addition fact 5+7=12.

5 + 7 = 12
12-5= 7; 12-7=5

Figure 1: A single addition fact

Layer 2: Process-object understanding of a number bond string

The next stage involves working with the set of decompositions of one number, e.g. 12. By string I refer to the complete set of possible number bonds or decompositions for 12, such as 12+0=12; 11+1=12. Being able to do the decompositions of each one separately implies a process understanding.

12+0 =12
11+1 =12
10+2 =12
9+3 =12
8+4 =12
7+5 =12
6+6 =12

Figure 2: The number bond string of 12

An object understanding involves seeing this set of possible decompositions as a whole or a totality, which enables comparisons of the elements making up the string, such as identifying relationships between the addition facts comprising the string, by de-encapsulation. An object understanding of each string facilitates further operations and manipulations on these strings, some of which are detailed in the next few paragraphs.

Layer 3: Using the encapsulation of the individual number bond strings to recognise patterns across them

When a concept has been encapsulated into an object, the learner can: de-encapsulate an object, thereby returning to the process in a form prior to its encapsulations.

De-encapsulation enables the learner to use the properties inherent in the object to perform new manipulations upon it (Meel, 2003: 152).

When the encapsulation of individual number bond strings is extended to strings of different number bonds (e.g. 12 or 13), a learner should be able to pick out particular decompositions across different strings, for example, seeing that 7+5=12, while 7+6=13 (together with the associated subtraction relationships).

When they can simultaneously see the difference between groups of sums such as 5+7=12, while 5+8=13, and 5+9=14, they are operating on the single strings as objects. By seeing 5+9=(5+8)+1 they are displaying an object understanding of the different individual strings, because they are able to see the differences between the sets of decompositions belonging to 13 or 14. For example, in the figure below, which shows the decompositions for strings 12 and 13, a child should be able to see a commonality in the addition facts of 7 by identifying patterns in the sums 5+7=12 and 6+7=13 across the different strings. Similarly, de-encapsulation of pairs of strings will allow them to see commonalities in adding and subtracting other numbers, for example, 9.

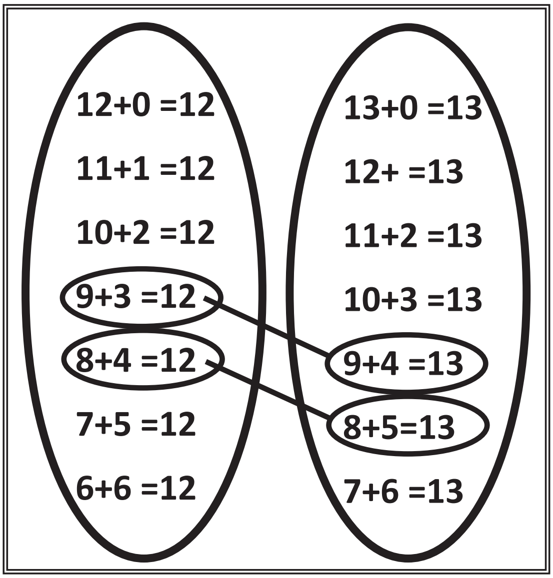

Figure 3: Recognising relationships across number bond strings

Further layers

Applying number bond strings in other complex calculations will also require a place value understanding, and this will allow them to work towards problems such as 25+8 and 35+8. For example, we can infer an object level understanding of the number bond string of 13 when they can see 25+8 as 20+5+8=20+13=33, showing that they are able to de-encapsulate the constituent elements of the individual number bond string of 13. Their understanding of place value has enabled them to break down the 25, while their object level use of the number bond string of 13 is revealed by replacing 5+8 by 13.

Multiplication layer

At the stage of working out single multiplication elements, they are able to draw upon seeing 3 lots of 7 dots and then seeing 4 lots of 7 dots as adding another 7 to the previous product. Thus, they could develop fluency in the multiplication facts of 7 by moving through 7; 7+7; 14+7; 21+7; etc. Fluency at this stage requires the previous encapsulations of the number bond strings.

Results

In this section, I provide a vignette of Lizzy's progression in addition and multiplication.

Multiplication by 7 and 8

Towards the end of Lizzy's Grade 5 year, I watched her struggle with multiplication of the kind 70×2 or 30×70 and found that she could not see 30×70 as 3×7×100. While investigating this, I found that she was using her fingers to work out the multiplication tables.

I was puzzled at the fact that, as soon as she tried to verbalise the tables, she reached immediately for her fingers. As I watched Lizzy, I realised that she was using her fingers as a positional reference; the first finger represented 7×1, the second finger was 7×2=14, etc. Then she would write down the sequence. Once when I asked her to recall the seven-times table she said: '7, 14, 22, 22, 21? I struggle to ever get that right'. This struggle showed that she did not recognise a pattern of addition by 7. It took many attempts of working through the various products before she could fluently move through the multiples of 7 as '7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84'. She could then generate in writing the pairwise products in order 7×1=7; 7×2=14; etc.

However, when questioned out of that order, I noticed that she could calculate some of them, but not all, for example, she was unsure whether 6×7=48 or 7×7=48. When challenged, Lizzy would go back to counting on her fingers, 1st 7, 2nd 14, and so on, and drawing on her two toes to help get to twelfth product. When asked if she confused 7×7 with the six-times table, she said no: 'I confuse it with this one [pointing to 4×7=28]'. This showed confusion between 48 and 28 (without realising that 7×7 is actually 49, not 48, in any case). After rehearsing the seven-times table, she would get them without a problem, and similarly for the eight-times table. However, as soon as I tried to go across the tables, she would get a bit confused when facts from both were required. This indicated a process understanding of each time table string, because she was able to work within one string at a time, but was unable to move across two or more strings.

I pondered about how I could help her move to a more fluent understanding of each of the tables. Her primary referent understanding of the multiplication facts of 7 was the use of her fingers, where she associated the different multiples with positions on her finger. I realised, however, that the automaticity of her multiplication depended on the previous layer of adding by 7. Thus, it seemed that a previous encapsulation of the strings of number bonds was not readily available to her. Lizzy's situation seemed to exemplify Simon et al.'s (2004) argument that 'if a certain conception is not available, the learner cannot recognise a situation' in the same way that someone who has the conception might. Lizzy could not readily access the addition and subtraction facts related to 7. I realised that I needed to help her find a concrete referent that could help her concretise the process of adding by 7. That is, I needed to help her to find a concrete embodied understanding of adding by 7.

I was spurred on to this task when I watched her struggle one day with the eight-times multiplication string. When asked to write out the eight-times table, Lizzy just wrote the answers 8, 16, 24, 32, … (as opposed to writing the fact 8×1=8; etc.) and explained: 'I knew the answers so I just wrote them down'. To generate the next number she just added the 8. When questioned she said 'I know it'. However, when she got stuck, around the 50s or 60s, she found herself in a tight position. When adding the next 8, she needed to free up her fingers from keeping place or position of the number that was multiplying the 8 in order to use her fingers to add the 8 again. At this stage, her anxiety deepened while she was trying to juggle the different processes that need to be carried out. When that was done, it meant starting counting in 8s again, but one step further with the new sum she had worked out. With all these different operations going on, trying to keep track of the different processes became harder to manage as she moved on to higher multiples of 8.

Anghileri (1989) presented a similar description of a learner's struggle to coordinate sequential processes by juggling the tasks between the fingers on each of her hands while trying to recount the three-times table:

> *Having counted the middle three fingers of her left hand, 'One, two, three' she raised one finger on her right hand. Now she focused again on her left hand to count, 'Four, five, six' and raised a second finger on her right hand. Using her left hand she counted, 'seven, eight, nine'. When she raised a third finger on her right hand, her gaze passed from right hand to left hand and back again. At this point, she abandoned her first attempt, apparently confused over the differing roles of the fingers on her right and left hands (Anghileri, 1989: 373).*

This description of trying to keep track of what was being counted and what was being multiplied, by assigning the functions to the fingers from different hands, resonates with the way in which Lizzy was trying to reproduce the times tables. The struggles of Lizzy and the learner in Anghileri's study (1989) demonstrate that, when a previous encapsulation has not taken place, further developments that rely on that necessary encapsulation become compromised and more complicated, leading to anxiety. However, if addition by 7s and addition by 8s was readily available to Lizzy, then the multiplication by 7 and 8 would be more manageable, because the automaticity of the addition would leave her some space to focus on the seven-times table. I realised that, if she could encapsulate addition facts for particular numbers (such as 1+7, 2+7, 3+7), this may improve her fluency in obtaining the multiplication facts for 7.

Problems with previous non-encapsulations

At this stage, I decided to go back to investigate Lizzy's use of addition facts. Lizzy had no problems seeing the different possible decompositions of the single addition facts such as 5+7=12. She was able to answer questions such as '5+what = 12' as well as '12–5 = what?'

In assessing her use of the number bond string, I first asked Lizzy to write bonds of 12 such as 0+12=12; 1+11=12. When she got to 5+7=12, she said: 'Then what comes next, I forgot'. I prompted her by saying '6+ what =12? She then completed the list and was able to recite the whole string, by rote. When asked to write the possible decompositions of 13, Lizzy proceeded without any problems. After she got to '6+7=13', she wrote, '7+8=13'. When asked 'what is 6+7=?' (reminding her of the previous step), Lizzy replied: 'Oh, I made a mistake'. She then corrected her error to write '7+6=13'.

She was able to rattle off the number bonds of 13. Also when asked questions such as '7+ what =13?'; '10+ what =13?', she was able to answer these without a problem. With questions such as '13–4 = what?', '13–7 = what?', she was a little bit slower but able to get to the correct answer. However, when I moved back to the question '12–7 = what?', she said '6'. This showed that she was only fluent when considering one string at a time. I then probed her by asking 'what is 7+6?'; her reply was 13. Then I asked: 'So what is 13 – 7'; the reply was '6'. So when I returned to the original fact (12–7) and asked her '7+ what is 12?', Lizzy was then able to reply correctly '5'. This shows that moving across different bonds by looking at addition by 7 or subtraction of 7 was staggered and not fluent.

Lizzy was working with the string of number bonds on a process level. She used a pattern starting with 0+12=12. She then increased the first number and decreased the second number to generate the new addition fact or decomposition of 12 (1+11=12). However, Lizzy has struggled to reach past this level – she could not see the string as a whole or a totality, which requires associating the different pairs of numbers in the string with 12. She could not identify those addition facts related to decompositions of 12, that is, she could not de-encapsulate the string of number bonds for 12. In order to use the different strings to fluently extract the addition facts of 7 across different strings (12, 13), one requires an object level understanding of these bond strings, which was not accessible to Lizzy.

An alternative route

In pondering over the barriers faced by Lizzy, I realised that the two years she spent learning the number bonds did not help her recognise pairs of numbers which add up to 12 or 13, say. She could see only those pairs in terms of the 12 or the 13 – she could not de-encapsulate the string which would allow her to 'use the properties inherent in the object to perform new manipulations upon it' (Meel, 2003: 152).

In order to gain fluency in multiplication by 7 (or by 8), she needed a fluency in addition by 7 (or by 8). Getting to that sense via the number bonds route, is complicated because it requires two layers of movement and no concrete embodiment to revert to, which could help consolidate or support her ways of working.

At this stage, I consulted with a colleague from Columbia I met at a conference, who suggested that I use two tables made up of ten cells each, as illustrated in figure 4, with red and black counters to represent different numbers. We practised addition facts of 9 such as 9+5=?; 9+6=?; where the 9 was represented by nine black counters in the first table, and the 5 or the 6 was represented by the red counters. The associated subtraction (15–9, say) could also be modelled by leaving the nine black counters on the first table and then placing in red counters until there were ten in the first table, then five in the second table. We then blocked the black counters, and asked: 'How many counters are left?'. This helped her see addition and subtraction by 9 in a concrete manner, and also helped her concretise various decompositions of 10.

Then, a similar situation was set up with eight black counters being used for addition by 8 and subtraction by 8. When she got stuck, she placed the counters and physically counted them. After a while, she could imagine what 8+7 meant – two reds to make up the ten, leaving five reds for the next block, therefore 15. This is referred to as the 'make ten' bridging strategy (Ernest, 2006: 90).

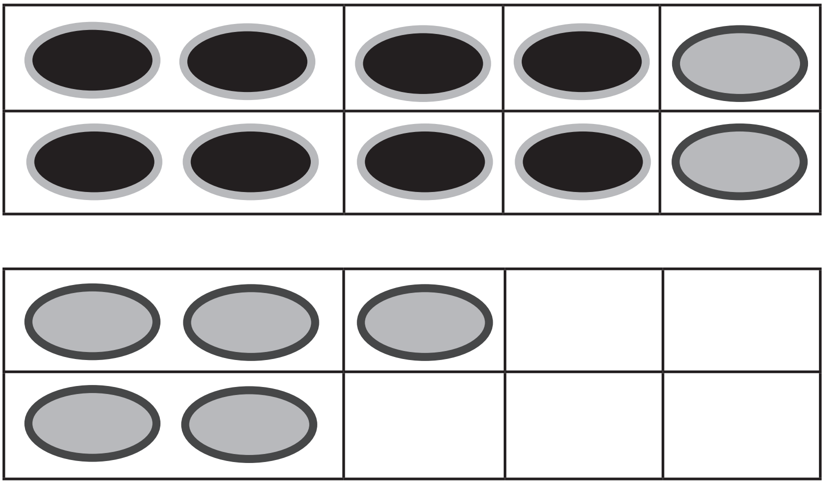

Figure 4: Using the 'make ten' strategy to consolidate addition and subtraction facts, Red counters illustrated as grey.

Ernest (2006) uses the example of 3+8 to argue that the sum involves a series of transformations which depend on an intuitive understanding of the associativity of addition. He lists these as 3+8, 3+(7+1), (3+7) + 1, 10+1, 11. However in this very concrete example, Lizzy could see that 8 +7 = 8+ (2 +5) = (8+2) +5 = 15, and the transformations inherent in the associative grouping were supported by the concrete situation. The sum 7 + 7 would be concretely modelled as 7 + (3 + 4) = 10 + 4, while 7 + 5 would be seen as 7+(3+2) = (7+3) +2 = 10+2 = 12. Thus, the use of the counters helped Lizzy access a concrete embodiment of what it meant to add 7 and another one-digit number. She first practised using the counters and then, when she was comfortable with that, she was able to reach the answer without physically moving the counters around. By doing this repeatedly, she could then see patterns in adding 7 to other one-digit numbers, leading her to an object understanding of the number facts associated with 7. De Lima and Tall's (2008) explanations of conceptual embodiment help shed light on this development. The manipulation of counters can be seen as a conceptual embodiment because it led her to see patterns in adding and subtracting by 7, eliminating the need for her to go back to checking with her fingers. Thus, using the counters for addition and subtraction by 7 acted as a conceptual embodiment allowing her to gain fluency in establishing the seven-times multiplication table.

Now that Lizzy had access to a conceptual embodiment of adding and subtracting by 7, her methods of working out the seven-times table (by using her fingers as position) worked fine for her needs. Lizzy's anxiety has reduced considerably. I think that having the concrete method has helped to make it less burdensome so that she only has to concentrate on finding the next multiple of 7. In time, she will reach the fluency in seeing the whole string of pairs making up the seven-times table. The increasing fluency has induced a sense of confidence; so she is more certain about the results she obtains. I have noticed that Lizzy is finding equivalent fractions and simplification of fractions much easier because the multiplication facts are now so fluent.

Discussion and concluding remarks

Lizzy's experiences illustrated that a previous non-encapsulation of the number bond strings hindered her from gaining fluency in multiplication strings which would, in turn, hamper her in further work that relies on multiplication and division. By using an alternative approach, she was helped to concretise her experience of adding and subtracting particular numbers such as 8 or 7.

Although Lizzy still uses her fingers as a prompt for the different multiples of 7, it is being used in a simpler manner because she does not have to interrupt the rhythm to then work out, for example, 56+7. If she momentarily hesitates, she has a picture that helps her 'see' the addition fact 7+6=13. The coloured counters served as a concrete embodiment that helped her develop a conceptual embodiment of addition and subtraction by 7 which is available to her now as an encapsulated object. As Ernest has noted, there are 100 one-digit addition facts, and a non-encapsulation of previous patterns adds a large load to the processes that children have to manage in the course of doing a calculation.

In primary schools, children spend an inordinate time (in Lizzy's case, two years) practising these number bonds or decomposition strings, because it is assumed that these form the building blocks for solving problems based on addition, multiplication and division in future. However, in Lizzy's situation all that effort spent on writing it down and rehearsing it was not productive, because it did not help her in building her understanding of the multiplication strings. A focus on the addition and subtraction by 7 and addition and subtraction of 8, etc., proved to be more useful to her in learning multiplication, because it opened up an alternative encapsulation path which was strengthened by the use of the conceptual embodiment (the counters).

Based on her experiences, I suggest that teachers consider using these addition and subtraction blocks, shown in figure 4, which focus on adding and subtracting a particular number. These can then lead to seeing patterns in adding and subtracting specific numbers. Trying to encapsulate the strings of bonds proved to be difficult, because we could not find a concrete embodiment that could be used to lead to a conceptual embodiment for further development. However, the use of the two blocks, which was based on the 'make ten' strategy, while being concrete, allowed Lizzy to go back to the board to verify and confirm her results. In this case, the concrete embodiment has facilitated further movement to a conceptual embodiment.

De Lima and Tall (2008: 17) argue that it is important to build upon the learner's experience, and they caution that a teacher has to be sensitive to the child's current knowledge and need to recognise 'earlier fragility of knowledge' as exhibited 'in subsequent knowledge that causes a rethink of how to teach earlier knowledge in future'.

In this detailed single case study, it was possible for me to introspect on many of these issues that De Lima and Tall (2008) have highlighted, because of the time that I was able to spend with Lizzy. Most teachers, however, do not have the luxury of providing such detailed attention. Nevertheless, there are many teachers who have developed insight through the teaching of similar concepts using similar methods, and many such dedicated teachers develop a repertoire of strategies and indicators which help them recognise which strategies may be more helpful than others. It is my hope that such teachers are supported and recognised for the work they do in reducing the struggles of their learners in learning mathematics concepts.

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