

The relationship between teachers' instructional practices and their learners' level of geometrical thinking

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This case study describes and investigates the instructional practices of Grades 1 to 5 teachers and the levels of geometry thinking of the learners, according to the Van Hiele model, with a view to determining whether there is a match between the instructional practice and the learners' level of thinking. The instructional practices of the teachers were observed and analysed, and their learners' levels of geometry thinking were accessed through a Van Hiele test. The results suggest that there is not a simple relationship between the phases of learning, as described by Crowley in 1987, and geometric development in terms of the Van Hiele levels. It is, however, possible to explain the geometric development to a limited extent in terms of the Van Hiele levels of the observed teaching activities. Although the presence of activities on an appropriate level does not guarantee growth in terms of the Van Hiele model, the absence thereof results in stagnation. The instructional practices in primary schools in all Grades should span geometry experiences on all the levels, because the pre-visualisation level and Van Hiele Level 1 thinking are still evident up to Grade 5.

Keywords: Geometry, Van Hiele theory, instructional practice, level of thinking, mathematics

Introduction

From 2014, Euclidean geometry will once again be made compulsory in the final examination of South African high schooling, having been made voluntary in 2008. However, according to the Curriculum and Assessment Policy Statement, the foundation of geometric knowledge and understanding starts in primary school (Department of Basic Education, 2011). The question is: What is the nature of the teaching and learning of geometry in primary school? The purpose of this exploratory case study was to plot the educational landscape of an independent, co-educational, primary school in Pretoria in terms of the geometry that is being taught and learnt in the Foundation and Intermediate Phases.

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Research suggests that mathematical knowledge is constructed relationally (Bransford, Brown & Cocking, 2000), re-enforcing the notion of building on learners' prior knowledge. It is essential that the learner's prior knowledge and prior experience are recognised and used as a foundation upon which to build new knowledge and skills (Van de Walle, 2007; Arzarello, Robutti & Bazzini, 2005). A model that recognised this principle was developed by Pierre and Dina Van Hiele, who proposed that geometry understanding develops in five sequential levels of thought. The Van Hiele model of geometry thinking was used as a framework for this study. This study is limited contextually, but carries implications in terms of policy and practice for other primary schools.

Literature review and theoretical framework

We integrate literature and theory in this section in order to provide readers with a sense of both the Van Hiele framework and its associated teaching activity models in relation to some of the aspects of geometry teaching that have been reported as problematic in previous studies. Battista (2007: 846) explains that “[a] considerable amount of research has established the Van Hiele theory as a generally accurate description of the development of students' geometry thinking”. The Van Hiele model has five sequential and discrete levels of thought. Each level is characterised not only by qualitatively different levels of thinking, but also by different internal knowledge and processing (Battista, 2007). Most significantly, the Van Hieles asserted that these levels of thinking are related to the types of geometry activities the learners have experienced rather than to the learners' physical development or maturity. The levels, as described by Mason (1998: 4-5), are as follows:

- **Level 1 (Visualisation):** Students recognise figures by appearance alone, often by comparing them to a known prototype. The properties of a figure are not perceived. At this level, students make decisions based on perception, not reasoning.
- **Level 2 (Analysis):** Students see figures as collections of properties. They can recognise and name properties of geometric figures, but they do not see relationships between these properties.
- **Level 3 (Abstraction):** Students perceive relationships between properties and between figures. At this level, students can create meaningful definitions and give informal arguments to justify their reasoning. Logical implications and class inclusions, such as squares being a type of rectangle, are understood.
- **Level 4 (Deduction):** At this level, students should be able to construct proofs such as those typically found in a high school geometry class.
- **Level 5 (Rigour):** Students at this level understand the formal aspects of deduction, such as establishing and comparing mathematical systems. They can understand the use of indirect proof and proof by contrapositive, as well as non-Euclidean systems.

There are five characteristics of the Van Hiele theory which have significant implications for instructional practice. The extrinsic/intrinsic characteristic explains that the inherent objects at one level become the objects of study at the next; thus, a curriculum to promote progression through the levels needs to exhibit a logical development in its content and processes. Arguably, the most significant of the properties is the sequential nature of the model. This characteristic, as explained by Usiskin (1982), implies that a student cannot be at Van Hiele level n without having gone through level $n-1$. Although the levels were considered by the Van Hieles to be discrete, other researchers have found these levels to be dynamic and continuous (Gutiérrez, Jaime & Fortuny, 1991). In the case of mismatch, instructional practices fall beyond the learners' level of understanding. For all intents and purposes, the teacher may just as well be speaking a foreign language (Atebe & Schäfer, 2008b). This may imply that students imitate the language used by their teachers who reason at a different level to them, with no real understanding. The resulting lack of understanding has been shown to be difficult to uproot and is widespread (Atebe & Schäfer, 2008b; Gerace, 1992), impeding further growth.

The Van Hieles (Crowley, 1987; Van Hiele, 1996) also proposed five *sequential phases of learning* or types of geometry experiences, which enable the learner to progress to a next level:

- **Phase 1 (Inquiry/information):** The teacher gains insight into the prior knowledge that the learners have about the topic, and the learners get an idea of the direction further study will take. Discussion and observations are characteristic of any activity in this phase. At this phase, questions are raised on observations, and level-specific language is introduced (Rudd, Lambert, Satterwhite & Zaier, 2008; Webb, Franke, Tondra, Chan, Freund, Shein & Melkonian, 2009).
- **Phase 2 (Directed orientation):** The teacher uses a variety of carefully sequenced short tasks to help the learners explore the structures characteristic of the level and to elicit specific responses (Chard, Baker, Clarke, Jungjohann, Davis & Smolkowski, 2008).
- **Phase 3 (Explication):** The teacher assists the learners in using appropriate and accurate language. At this phase, the learners verbalise and express their thinking and observations about the topic (Webb et al., 2009). In this instance, the level's system of relations starts to become apparent.
- **Phase 4 (Free orientation):** Learners explore relations within the level or 'field of investigation' so that the relations between the objects of study become explicit to them. The teacher facilitates this process by presenting the learners with multi-step tasks, tasks with several means of solving them, and open-ended tasks (Cobb, Wood, Yackel, Nicholls, Wheatley, Trigatti & Perlwitz, 1991).

- **Phase 5 (Integration):** In this phase, it is important that no new information be presented, but that the learners summarise and review what they have learnt in order to form an overview of the objects and relations they have investigated. The teacher's role in this phase is to ensure that a complete (relevant to the level) summary is formulated and the origin of this summary is reviewed (Gerace, 1992).

Research methodology

This study may be classified as an exploratory case study that seeks to chart the 'geometry terrain' within a private primary school in Pretoria. Learners' levels of geometry thinking were assessed halfway through the academic year in 2010 and all the other relevant data were collected within a six-month period. The study population spanned Grades 1 to 5. In order to determine the instructional practices of the teachers, their lesson plans were examined in conjunction with an observed lesson. The teachers in each Grade plan the tasks of their lessons together; therefore, a random selection of one teacher per Grade for observation was deemed acceptable.

The Cognitive Development and Achievement in Secondary School Geometry Van Hiele Geometry Test, developed by Usiskin (1982), was used to determine the learners' levels of thinking. During administration of the Van Hiele tests in the Foundation Phase, the questions were read aloud to the learners so that neither the mathematical language nor their reading ability inhibited their understanding of the questions. The assessment was introduced to the learners as an activity, not as a test, and they were encouraged to ask questions if they were unsure of what to do. The Van Hiele Geometry Test consists of twenty-five multiple-choice test questions (five questions per level). The criterion that was applied in this study worked on a 60% pass rate per level. Classroom observations were video recorded and deductively analysed using Crowley and Van Hiele thematic categories as per the theoretical framework.

Results

The learners' levels of thinking are presented first, followed by a juxtaposition of the observed geometry experiences and phases of learning per Grade. From Table 1 and Figure 1, it seems that progression through the levels of thinking from Grade 1 to Grade 3 is limited. The number of learners thinking at the Visualisation level increases from Grades 1 to 2 and from 2 to 3 by 6.2% and 4.9%, respectively. We notice a more drastic progression through the levels in Grades 4 and 5. Pre-visualisation is a term used in literature to describe learners who are not yet thinking at Van Hiele Level 1.

Table 1: Learners per Grade on each Van Hiele level

TVHT 60%	Level 0: Pre-visualisation	Van Hiele Level 1	Van Hiele Level 2	Van Hiele Level 3
Gr. 1 (N = 47)	25 (53.2%)	22 (46.8%)	0	0
Gr. 2 (N = 49)	23 (46.9%)	26 (53.1%)	0	0
Gr. 3 (N = 50)	21 (42.0%)	9 (58.0%)	0	0
Gr. 4 (N = 58)	10 (17.2%)	33 (56.9%)	15 (25.9%)	0
Gr. 5 (N = 46)	7 (15.2%)	20 (43.5%)	13 (28.3%)	6 (13.0%)

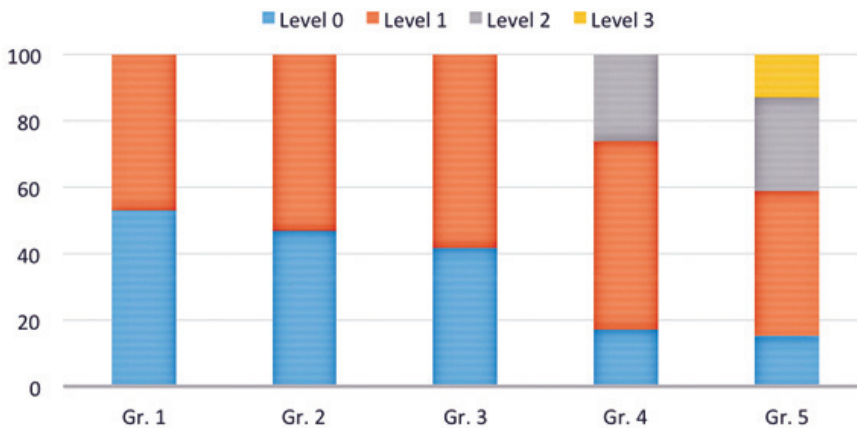


Figure 1: The percentage of learners' levels of geometry thinking per grade

There were some anomalies in the results: three learners in Grade 4 and two in Grade 5 scored below 60% on Level 1, but then scored 60% or higher on Level 2; however, the findings provided sufficient data of the predominant levels of geometry thought of the learners in these Grades. In the light of this study, the results indicate that instructional practices for the development of geometry reasoning should predominantly focus on the activities of Van Hiele Levels 1 and 2.

Grade 1 results

Table 1 and Figure 1 show that, in Grade 1, at least two Van Hiele levels of thinking are evident. Whilst 22 of the 47 learners in Grade 1 operated on Van Hiele Level 1, 25 of them were not yet there.

The observed lesson started with a class discussion about the three different cut-out shapes the teacher was holding. She asked the learners if they could remember what it meant to compare things. There was consensus that it meant to look at *'what was the same'* and *'what was different'*. The teacher held brightly coloured cut-outs of a triangle, a circle and a rectangle. Without naming the shapes, she asked the learners to compare them. The learners commented on the number of sides and the

number of corners. This type of activity is appropriate for Level 1 at which thinking focuses on appearance of shapes and not reasoning. It also aligns with the first phase of learning (Crowley, 1987; Van Hiele, 1996) in that it allows the teacher to gain insight into learners' prior knowledge and the learners get an idea of the direction of further learning. The following excerpt seems to indicate the third phase of learning in which the teacher's role is to assist learners in using appropriate and accurate language; however, the five phases are sequential and the second phase, in which short tasks are used to direct learning, was not observed in this lesson and thus the excerpt indicates Phase 1 learning.

Teacher: When you're talking about sides, do you mean these things here?

(Teacher indicated the edge of the shape with her finger). Why don't you think a circle has any sides?

Learner: A circle has no sides because it is not straight.

Teacher: So does a circle have sides? Yes, a circle has one side but it is curved.

What are these? (Teacher touched the three angles of the triangle)

Learner: Corner is sharp pointy.

Teacher: What happens at a corner?

Learner: Two sides join!

The teacher then produced a model of a cylinder, a cuboid and a triangular prism and repeated the discussion. The learners then had to compare the two-dimensional shape with its corresponding three-dimensional object and record their findings. This task profiles the fourth phase of learning in which learners are to explore relations within the field of investigation. Learners were not given the opportunity to review and summarise their learning, hence there was no opportunity for the fifth phase of learning.

Grade 2 results

Of the forty nine learners, twenty three were unable to recognize a square when presented in a different orientation or identify a triangle accurately or distinguish a square from a rectangle. This is an indication that these specific learners were not yet functioning on the Visualization level at which a figure is identified by what it looks like. Table 1 and Figure 1 show that 26 learners were, however, able to do this and therefore seemed to operate at Van Hiele Level 1.

The Grade 2 lesson started with the teacher asking the learners to draw a house using only two shapes on their whiteboards. The teacher prompted discussion by asking what shapes could be seen in each learner's picture and how the pictures were similar or different. The recognition of shapes by appearance alone represents a Level 1 activity but was only used as an introduction. After the introduction, the teacher produced a model of a regular polygon and used it to prompt the next drawing task. Throughout the drawing activity, the teacher would use the learners' responses to

draw attention to the properties of the various shapes being used, which presents Level 2 thinking. The following excerpt is from the class discussion:

Teacher: What are the rules for a square?

Learner1: It has to have four sides.

Learner 2: The sides have to be the same.

Teacher: What are the rules for a triangle?

Learner 3: It has to have three sides. (Showed his picture of what looked like a scalene triangle to which another learner said that a triangle has to have sides that were the same)

Learner 4: No! They don't have to be the same size, it can have two long and one short.

Teacher: But can they all be the same size? ... Yes. What are the rules for a rectangle?

Learner 5: It must be two long and two short.

This activity allowed the teacher to gain insight into the learners' prior knowledge and gave direction to the learning to come and so falls within the description of the first phase of learning. She also gave short tasks designed to elicit specific responses aimed at making explicit certain properties of the shapes used, indicative of the second phase, Directed orientation. The teacher ended the class discussion by assigning different tasks to the three differentiated groups. Two groups had to complete a worksheet at their desks, while the third built objects with geo-structa pieces at the back of the classroom. The worksheets required the learners to operate on Van Hiele Level 2. The learners observed and recorded the different properties of shapes either by counting the number of sides/corners or by drawing. The group tasks gave opportunity to Free Orientation, Phase 4, but neither the third nor the fifth phase of learning, Explication and Integration, were included in this lesson.

Grade 3 results

Table 1 and Figure 1 show that 21 of the fifty Grade 3 learners were not yet operating on Van Hiele Level 1. Twenty-nine learners were able to consistently and accurately recognise a shape based on its physical properties.

The Grade 3 lesson was the third in a three-day rotation in which 3 groups were given one of three tasks on Van Hiele Levels 1 and 2 to complete each day. The lesson started with a discussion directed by the teacher about what had been learnt during the previous days. This enabled the learners to reflect on what they had learnt and how they had gone about each of their three tasks:

- Learners had to construct a three-dimensional object, in this case a house with a square base and a triangular prism for a roof, with straws and Prestik. The learners were given a task sheet with the written

objective and dimensions clearly typed. There was no diagram on the task sheet.

- The learners were given a task sheet asking them to design a floor tile of given dimensions using Cuisenaire pattern blocks. They were asked to ensure that their design had at least two lines of symmetry and they had to repeat the pattern three times.
- The learners worked together in pairs against the clock to complete a Tangram puzzle. The teacher moved from group to group checking on progress and prompting learners. There was consistent interaction among the learners concerning their tasks. The opportunity for the learners to explore shape, albeit implicitly, is appropriate for learners acquiring and consolidating Van Hiele Level 1 thinking. These activities are multi-step tasks with more than one solution, allowing the learners to explore relationships between shapes and, therefore, developing Van Hiele Level 2 thinking. These are the kind of tasks identified in the fourth phase of learning.

Grade 4 results

Table 1 and Figure 1 show that of the 58 learners, 10 did not yet consistently operate on Van Hiele Level 1. Thirty-three learners could be described as operating on Van Hiele Level 1 (Visualisation) and 15 learners seemed to be able to operate on Van Hiele Level 2 (Analysis).

The Grade 4 lesson started with a multiple-choice-type group quiz. The teacher then presented a worksheet to each group and asked them to match the shape with its name; this is a typical Van Hiele Level 1 activity. The teacher asked the learners to make a right angle using their arms and then asked them similarly to show an 180° angle. The teacher then called on a volunteer to help her demonstrate the meaning of a revolution and half a revolution. The teacher drew a circle on the board to represent a 360° protractor and defined an angle as 'the space between two lines that meet'. Whilst repeating this statement, the teacher traced the line between the markings on the circumference of the circle she had drawn on the board. Discussion, characteristic of Phase 1, among the learners during the quiz allowed the teacher to assess learners' prior knowledge as were the actions she got the learners to perform. There was no significant evidence of any other learning phase. Since the learners were required to name the shapes according to their appearance, this lesson was at an appropriate Van Hiele level for these learners.

Grade 5 results

Table 1 and Figure 1 show that of the 46 Grade 5 learners, 7 were not able to consistently and accurately identify a triangle or differentiate between a square and a rectangle, implying that they were yet to operate on Van Hiele Level 1. Twenty

learners seemed to operate on Van Hiele Level 1 and 13 were able to successfully answer three or more of the five questions posed at Van Hiele Level 2. Six learners were also able to answer three of the five questions posed at Van Hiele Level 3.

The two weeks of geometry teaching reflected in the planning began with a discussion to revise names of known shapes and introduce new shapes. Textbook exercises on angles and tessellations as well as basic calculations of area and perimeter were also included during this time. The teacher started the Grade 5 lesson with a Van Hiele Level 1 activity by asking the learners to write down in three minutes all the shapes they knew with straight lines. The learners counted the names of figures they had listed and, after some discussion, the learner with the longest list was congratulated as the winner. This activity is indicative of Phase 1: Inquiry/information. The teacher then produced colourful posters of various unnamed polygons and asked the learners to name the shape, giving a reason for their answer (Van Hiele Level 2 activity). During this activity, the teacher guided the learners to use the correct terminology, sometimes explaining unfamiliar words. The teacher also interrupted the activity to probe the learners' understanding in terms of the differences between a square and a 'diamond' (rhombus). This probing of relationship between different figures develops Van Hiele Level 3 thinking. The teacher's actions exemplify the role of the teacher in phase 3 in which s/he is responsible to help the learners use relevant and accurate language. The class session ended when the teacher explained the group project which was to be done and handed task cards to groups of five learners. Each group was commissioned to produce an informative poster and a mobile focusing on a particular shape and a three-dimensional prism. The learners were permitted to use any resource at their disposal, including their dictionaries and textbooks. Discussion centred on the technicalities of the task or the definitions of the figures and prisms. In this task, learners were required to review their knowledge and present a summary.

Discussion

Table 1 and Figure 1 show that, in Grade 1, at least two Van Hiele levels of thinking were evident. By implication, instructional practices to develop geometry understanding should encompass the range of learners' levels in that Grade. This means that the learners require multiple and diverse activities to enable them to progress through the levels of abstraction (Battista, 2007) and hence identify the characteristics of certain shapes for them to begin to conceptualise classes of shapes. Learners need the opportunity to progress through all five phases of learning in order for the inherent characteristics of these shapes to become explicit and hence become the objects of study in the next Van Hiele level. The lesson observed at this Grade level presumed a level of understanding which was not uniformly the case. For example, in linking the two-dimensional shapes with three-dimensional objects, the teachers presume that the learners are able to identify a triangle and a square in any orientation (Van Hiele Level 1). In making this assumption, their instructional

practices fall beyond 53.2% of the learners' levels of understanding. In neglecting the sequential nature of the phases of learning, in the same lesson, the teacher did not give the learners the opportunity to construct their own meaning through directed short tasks (Phase 2), nor was there an opportunity for the learners to review and locate this understanding in a conceptual schema (Phase 5). Thus, the probability of mismatch increases as the teacher is likely to presume that the learners have gained understanding which is unlikely, even though the teachers made use of examples of shapes and objects that were familiar to the learners.

The Van Hiele level at which the Grade 2 lesson was presented did not match the learners' thinking level. All the activities, except for the introduction, centred round the properties of shapes (Van Hiele Level 2), although 46.9% of the learners had not yet reached Level 1, thus indicative of mismatch. The activities in the Grade 2 lesson were in alignment with the second phase that Crowley calls Directed orientation. The frequent use of group work and the indication of specific vocabulary to focus on in that particular lesson suggest that Crowley's third phase of learning, Explication, was also included in Grade 2 geometry lessons. There is, however, no indication of the fourth phase of learning being accommodated. Free orientation, where the learners are provided with open-ended, multi-step tasks was not observed. This type of activity is the kind of task that enables learners to think geometrically.

The Grade 3 teachers developed Van Hiele Levels 1 and 2 thinking which is appropriate, because 58% of the learners are functioning on Van Hiele Level 1 and the remainder on Van Hiele Level 2. These teachers also seemed to use differentiated group work regularly. Placing the learners into smaller groups of like ability allows for a greater opportunity to teach within each group's particular level of understanding. The structure of the classes and the appropriate level of the lessons in Grade 3 provided opportunity for the development of geometry insight, as may be deduced from the significant number of learners that move up a Van Hiele level from Grade 3 to Grade 4. This happens despite the fact that only the first two phases of learning, as described by Crowley (1987), were observed.

In limiting the activities in Grade 4 to only some of the phases of learning, this planning inhibits the learners from integrating the ideas presented to them with their existing schema and from developing greater insight into geometry concepts. Although there were some learners yet to function at the Van Hiele Level 1, the predominant levels of learners' geometry thinking are Visualisation and Analysis, meaning that the instructional practices at this Grade level should accommodate a wide range of understanding. The activities only partly reflect those suggested in research for Level 1. Evident in the classroom observations were the use of different teaching strategies and the use of manipulatives.

The Grade 5 lesson demonstrated activities on Van Hiele Levels 1, 2 and 3, which is appropriate for the majority of the learners, according to Table 1. There was, however, no evidence of activities that require justification and develop the ability to prove for the 13.0% of the learners who already function on Level 2. The first three

phases of learning suggested by Crowley (1987) for the development of geometry reasoning were evident. There was no convincing evidence of the last two phases of Free orientation and Integration being included in the instructional practices at this Grade level. The property of Fixed sequence becomes relevant. During the class visit, manipulatives were used, learners were actively engaged in the learning process and the group work observed afforded the opportunity for learners to negotiate meaning and for teacher-learner dialogue.

Conclusion

The purpose of the study was to describe the geometry teaching and learning at a specific school in relation to the Van Hiele model. The Van Hiele test results (obtained half way through the year) in Grades 1 to 3 show very little movement in numbers from a pre-visualisation level to Level 1. The test scores report 24.7% more learners in Grade 4 on Level 1 thinking or higher than those in Grade 3, whereas 13% of the Grade 4 learners developed and managed to operate on Van Hiele Level 2 in Grade 5. However, the teaching in terms of the sequential phases of learning, as described by Crowley (1987), was limited in all the Grades. This includes Grades 3 and 4 where the shift in the Van Hiele levels occurred. It is, therefore, not possible to state a simple relationship between the phases of learning, as described by Crowley (1987), as observed in the classroom, and geometric development in terms of the Van Hiele levels.

It is also not always possible to explain the geometric thought development or the lack thereof in terms of the level of the teaching and learning activities of the observed lesson. Although the level of the activities in the Grade 1 classroom was well aligned with the learners' level of thinking, there was almost no improvement in terms of the levels. It may be possible that the development of Van Hiele Level 1 requires more time than the other levels. Gutiérrez et al. (1991) argue that the acquisition of a specific level can take months and even years, but do not explain which levels may take more time. The small improvement from Grade 2 to Grade 3 is understandable, because in Grade 2 the teaching created a mismatch for 23 of the 49 (46.9%) learners' levels of thinking. The Grade 3 teaching activities covered both Van Hiele Levels 1 and 2, which is appropriate for all the learners. This might be a reason for the improvement in the Van Hiele test scores from Grades 3 to 4. Furthermore, the Grade 4 teaching involved Van Hiele Level 2 and higher activities, although 17.2% of these learners were not even operational on Level 1. This may be a reason why 15.2% of the Grade 5 learners had not progressed beyond Van Hiele Level 0. Although the presence of activities on an appropriate level does not guarantee growth in terms of the Van Hiele model, the absence thereof will result in stagnation.

The instructional practices should span the range of these levels and be relevant to the varying degrees of acquisition of those learners being taught. If we do not meet the learners on their level of thinking, the learners may imitate the reasoning and language of a Van Hiele level presented to them, but may not be able to successfully

appropriate or apply this reasoning to new contexts (Atebe & Schäfer, 2008a). Once again, these learners are at risk of stagnating at a particular level with each academic cycle in which the geometry curriculum is presented at a level incongruent to their level of understanding.

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