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Pedagogical implications of students' misconceptions about deductive geometric proof

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The role of proof in school geometry has been a subject of intense debate throughout the twentieth century and that debate persists even today. This study aims to identify and analyse deductive geometric proof difficulties encountered by Bachelor of Education (BEd) in-service student teachers and to propose possible ways of remediating them. The authors conducted a content analysis of responses to a circle geometry item in an achievement test taken by 170 students. Although 78% of the students performed well in the deductive proof item, 22% evidenced misunderstandings or misconceptions which varied in complexity. The misconceptions were analysed into four categories and implications for pedagogy proffered to turn the misconceptions into critical teaching and learning opportunities.

Pedagogiese implikasies van studente se wanbegrippe oor meetkundige bewysvoering

Die rol van bewysvoering in skoolmeetkunde was 'n onderwerp van intense debatvoering gedurende die twintigste eeu en gaan vandag nog voort. Die doel van hierdie studie is om probleme wat onderwysers in die Baccalaureus Educationis (BEd) met deduktiewe bewysvoering in meetkunde ondervind te identifiseer, te ontleed en om voorstelle te maak vir remediëring. Die skrywers het die inhoud ontleed van onderwysers se antwoorde op 'n vraag in sirkelmeetkunde in 'n toets wat deur 170 onderwysers geskryf is. Hoewel 78% van die onderwysers goed gevaar het met die deduktiewe bewys, was dit uit 22% van die response duidelik dat hulle wanbegrippe of wanopvattings het wat wissel in kompleksiteit. Hierdie wanbegrippe is ontleed en in vier kategorieë opgedeel. Voorstelle word gemaak van die pedagogiese implikasies om hierdie wanbegrippe reg te stel en te omskep in waardevolle onderrig- en leergeleenthede.

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In the discipline of mathematics, very little is regarded as highly as proof, and by that logic the learning of proof has been a major goal of mathematics curricula in many countries and for many generations (Harel & Sowder 1998, Otton 2007).[1] Otton further attests that it is proof that traditionally sets mathematics apart from the empirical sciences. The pre-eminence of proof is corroborated by Chazan (1993) who argues that it is one of the characteristics of mathematics responsible for the central role the subject plays in the humanities, the sciences, the world of work, and the world of man in Western thought. Within mathematics proof is regarded as an essential tool in understanding the mathematical thinking and reasoning used in the development of new concepts, verifying mathematical statements and showing their universality (Gfeller 2010, Hadas *et al* 2000). For Hadas *et al* (2000) the two classical reasons for teaching proofs were to teach deductive reasoning as part of human culture, and to serve as a vehicle for verifying and showing the universality of mathematical statements. That universality presumably allows one to distinguish clearly between fact and fiction, truth and falsity. As a consequence of the privileged position of proof within and beyond mathematics, students' and teachers' understanding of this mathematical method has been studied extensively both globally and locally.[2] It might be expedient at this stage to distinguish between deductive and inductive reasoning.

1. Historical background of the role and meaning of proof

1.1 Deductive versus inductive reasoning

Traditionally "proof" in mathematics has asymmetrically meant "deductive proof" in geometry. As a result the idea of proof, as a deductive process, in particular, has historically been stressed in the

1 The study was funded by Tshwane University of Technology. The opinions expressed are those of the authors and do not necessarily reflect the position, policies or endorsement of the University. We thank the anonymous reviewers for their valuable comments and suggestions.

2 *Cf* Balacheff 1988, De Villiers 1999, Gonzalez & Herbst 2006, Hanna 2000, Herbst 2002a & 2002b, Mudaly & De Villiers 2004.

teaching of geometry but not in the teaching of algebra or arithmetic (Harel & Sowder 1998). Deductive proof (logic or reasoning) must, in turn, be distinguished from inductive proof (logic or reasoning), which is its antithesis that mathematics shares with the sciences. By deductive proof in this article, we mean drawing conclusions from a logical chain of reasoning in which each step follows necessarily from the previous (Simon 1996). By inductive proof we contrastingly mean the drawing of a generalised conclusion from particular instances. In another sense, deductive proof is reasoning from the general to the particular while inductive proof conversely refers to reasoning from the particular to the general.

1.2 Justification for the inclusion of geometry in the curriculum

The justification for the inclusion of deductive proof in the school geometry curriculum has had a tumultuous history primarily in the twentieth century. Gonzalez & Herbst (2006) meticulously narrate and analyse the historical evolution of arguments for the geometry course in the curriculum. In their chronology, they identify four modal arguments constituting the locus of conflict. The first argument is that geometry provides an opportunity for students to learn logic with a transfer value to other domains (the formal argument). Hoeflich (1986: 96), for example, makes the claim that law can be reduced to a set of first principles, on the order of mathematical axioms, and that by the use of the deductive method, these principles, as in deductive geometric proof, can yield all necessary consequences. The second modal argument is that geometry allows connections to the real world if students' experiences are matched with the demands of their future careers (the utilitarian argument). Watts (1994: 5), for example, makes the claim that knowledge that fails to impress itself on everyday life in some way or other is untethered, irrelevant, easily surrendered and easily lost. The third modal argument is that proof affords students experiences that resemble the activity of mathematicians (the mathematical argument). Freudenthal (1971 & 1991), for example, makes the claim that a guided reinvention approach fosters an attitude of experiencing mathematics as a human activity, rather than as a set of frozen results bearing no traces of the activity by which they were created. The fourth and final modal argument is that geometry provides students with a unique opportunity to apply the intuition of the geometric objects in describing the world (the intuitive argument). Gonzalez & Herbst (2006), for example, make the claim that the geometry course can provide students with a language that could allow them to model the world.

1.3 The unsuccessful teaching and learning of proof

Despite the multiple justifications for curricula inclusion and the multiplicity of studies about proof learning, problems of students' understanding of deductive geometric proof continue to recur (Hadas *et al* 2000, Recio & Godino 2001). Such a recurrence suggests that efforts should not be spared until we can effectively and creatively teach for the clearer understanding of proof by a greater number of students. In this connection, Clements & Batista (1992: 55) point to the irony that, although there have been numerous attempts to improve students' formal proof skills by teaching proof in novel ways, nearly all have been unsuccessful. For Usiskin (1980) and Ernest (2007) we seem to have failed in our teaching of proof simply because we often ignore when and why mathematicians do proofs, that they do not write proofs in two columns, but spend most of their time exploring and conjecturing, and then systematise their results as a drawn out last step. This observation resonates with Freudenthal's (1983) antididactical inversion of the order in which mathematics has traditionally been taught.

Regarding the situation in South Africa, Wessels (2001: 3) unequivocally concedes that Euclidean geometry education is a complete disaster ostensibly because it is badly taught. Van Niekerk (1997: 112) categorically ascribes the failure to transform geometry instruction in South African schools to the fact that "... the majority of mathematics teachers are poorly trained". Similarly, Fish (1996: 8) laments that not all teachers are sufficiently competent to teach even the mathematics prescribed in the current syllabus. That includes geometry which in the recent past had to be removed from the mainstream mathematics curriculum and offered as an optional paper at the National Senior Certificate level. It is obvious that teachers cannot effectively teach topics with which they themselves are uncomfortable. That which teachers do teach is arguably limited and proscribed by that which they can teach comfortably (Hobden 1998).

1.4 What can we learn from students' failures?

The 2006 Trends in Mathematics and Science Study (TIMSS) report notes that, although mathematics teachers in South Africa were the least qualified from among the fifty countries surveyed in 2003, they were among the most frequently in-serviced (Reddy 2006: xv). This could be an indication of how disturbingly low subject matter knowledge is among teachers of mathematics, which on its own can have deleterious consequences on pedagogy. As a contribution to the redress process, in this study we focus on some common misconceptions as typified by in-service student teachers' responses with specific reference to deductive geometric proof. We hold the assumption that different levels of misunderstanding may reveal possible ways of remediating problems experienced by the students. In supporting the rationale of studying students' misconceptions or preconceptions, Carpenter (1996) is convinced that wrong answers by students can become a point of departure for rich discussion about mathematics pedagogy. Indeed, as Fang (2010) similarly notes, wrong answers can engender a cultural pedagogy of transforming errors in geometric proof assignments into resources for teaching and learning of logical thinking habits from early on. Our hypothesis in this article is that it is also possible to use the mistakes or alternative conceptions students make or hold as a pivot to empower them in "proof checking and construction" which in themselves are a pedagogical strategy that has been shown to work effectively (Clements & Batista 1992: 55).

2. Theoretical and contextual background

2.1 Justification for the proof item choice

The 2006 TIMSS report alluded to earlier further shows that South African students performed worst in Mathematics (and Science) out of 50 participating countries and that the weakest area of performance was geometry (Reddy 2006). This suggests that geometry education requires urgent attention and hence this study is timely. The selected proof item was a multi-step proof problem amenable to multiple solution strategies. Susceptibility to multiple solutions is a feature encouraged by the Realistic Mathematics Education (RME) approach practised in the Netherlands, a country that performed very well in the 2003 TIMSS. The proof problem was therefore open-ended and allowed students free productions of their own solutions or proofs. Open-endedness is a crucial RME principle of instructional and assessment designs (De Lange 1995). The item thus afforded students the opportunity to re-invent mathematics through their own constructions and reconstructions as advocated by Freudenthal (1991). Freudenthal pioneered the RME approach in the Netherlands by encouraging, *inter alia*, a view of mathematics as a human activity in which students must engage.

The RME approach itself is, in turn, compatible with constructivism which insists that knowledge cannot simply be transferred ready-made from parent to child or from teacher to student but has to be actively built by each learner as the primary actor in his/her own mind (Anthony 1996, Glasersfeld 1995). In other words, the item selected allowed the student teachers to build their own proof independently. This article focused precisely on the characteristics of the proofs (mis) constructed by students. The complexity of a proof itself is described in terms of the number of arguments a student has to combine (Heinze *et al* 2008). In a multi-step proof the aim of the proving process is to construct a sequence or chain of arguments from X (given premises) to Y (conclusion) with supportive reasons or intermediate hypotheses. Thus the proving process is a simple (one-step) or hypothetical (multi-step) bridging of a given initial condition to the wanted conclusion (Heinze *et al* 2008).

2.2 The influence of the van Hiele theory of geometric thought development

It is also imperative to consider the van Hiele levels of geometric thought development which wielded tremendous influence on geometry education reform in the last half of the twentieth century. Van Hiele (1986) theorised that students progress through five levels of geometric knowledge: the visual, the descriptive/analytic, abstract/relational, formal deduction and ultimately, rigor/mathematical. The intuitive foundation of proof, or deductive reasoning, first occurs at the third level when students are able to make connections between networks of statements about properties of shapes and relationships between geometrical objects (De Villiers 1999, Van Hiele 1986). A major strength of the van Hiele theory is that it emphasises the scaffolding role of teaching and learning that leads students to progress from one level to the next, and to the next. Another critical strength of the theory is its recognition of the role of language. The neo-constructivist theory of conceptual change suggests that faced with having to absorb new material that is in some way incompatible with prior knowledge the students will try to assimilate new information into their existing framework (Kajander & Lovric 2010). The student experiences cognitive conflict when new information is received, and has to resolve that conflict by interpreting the new information in terms of the old in order to establish new conceptual networks and restore equilibrium in the Piagetian sense.

2.3 Expanding the goals of teaching proof in school mathematics

We also need to engage in some rethinking about the goals of teaching proof. As mentioned earlier, for mathematicians, proofs have conventionally played an essential role in establishing the validity of a statement and in shedding light on the reasons or premises that support that statement (Hadas *et al* 2000). These traditional goals of teaching proof have recently been extended and fleshed out considerably by mathematics educators. Closer to home, for example, de Villiers (1999 & 2002) has identified six functions of proof: verification (concerned with the truth of a statement); explanation (providing insight into why a statement is true); systematisation (the organisation of various results into a deductive system of axioms, major concepts and theorems); discovery (the discovery or invention of new results); communication (the transmission of mathematical knowledge), and intellectual challenge (the self-realisation/fulfilment derived from constructing a proof). This suggests that the language of proof has to be varied: from "prove that" to "show that", "demonstrate that", "verify that", "explain why", "how can you convince/tell a friend", and so on.

According to Battista & Clements (1995), educators continue to debate the relative importance that formal proof should play in high school geometry. While some argue that we should continue or resume the traditional focus on axiomatic systems and proof, some believe that we should abandon proof for a less formal investigation of geometric ideas, and yet others believe that students should move gradually from an informal investigation of geometry to a more proof-oriented focus. In this article we are convinced about the importance of proof learning, within and beyond mathematics, in spite of the difficulties learners encounter. We therefore concur with Dickerson & Doerr (2008: 408) that "... proofs help students to develop logical or critical thinking skills that are useful beyond the mathematics classroom". We add that the geometrical proofs students learn in high school can prepare them more effectively for higher education studies in the Science, Technology, Engineering and Mathematics (STEM) careers which invariably carry a high exchange value in the global knowledge economy. However, it remains critical, in the introductory years, to present proof in a non-threatening, less rigorous and more informal way. We are also not particular about the two-column proof format as a method of communicating or executing proof, if only to emphasise that every statement must be justified by a reason leading to sound and valid argumentation.

3. Research questions

By investigating the nature of difficulties students encounter we can make a contribution to improve the teaching and learning of deductive geometric proof. Accordingly, this study attempts to answer the following: What is the proportion of in-service student teachers failing to solve a deductive geometric proof problem? What patterns of misconceptions or misunderstandings about deductive geometric proof can we discern from the students encountering difficulties? How can these misconceptions be pedagogically managed to make the deductive geometric proving process potentially more accessible to students?

4. Methodology

4.1 Participants

Participants in the achievement test were 170 in-service student teachers enrolled for the Advanced Certificate in Education (ACE) at a South African university. Of these, 94 (55%) were females and 76 (45%) were males. In this ACE programme, students specialising in mathematics education typically take geometry education as one of their four courses intended to strengthen their understanding of school mathematics for effective teaching at the Further Education and Training (FET) band. This course is offered at the Higher Education and Training (HET Level 6) band. Participants were students whose language of instruction was English as a second or foreign language.

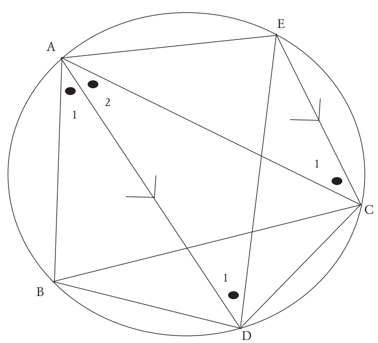
4.2 Test item selected for analysis

The item was selected to enable us to hypothesise about the student teacher's level of geometric thought development and thus serve as a compass for determining the nature and level of intervention.

Figure 1 shows the item with some angles marked/numbered to hint at equality and make alternative labelling possible. For example, $\angle A_1 = \angle BAD$, $\angle A_2 = \angle DAC$, and $\angle D_1 = \angle ADE$.

In the sketch AD//EC and ∠ BAD and ∠ DAC.

Prove that AB//ED.



4.3 Data collection and analysis

Only those proofs that were presumed to be incorrect or only partially correct were considered for further analysis. From that collection those examples which were typical of a particular pattern of responses were selected for discussion. Data collected were analysed both quantitatively and qualitatively. Table 1 shows the scoring rubric used for the assessment of students' responses.

Table 1: Scoring rubric

Mark allocated	Description of performance
0	Lack of basic geometrical knowledge and vocabulary or lack of appropriate geometrical frame of reference
1	Recognition of some helpful facts but inability to make a lo(gi)cal deduction
2	Ability to notice helpful facts and make an inference, but inability to organise information in logical (efficient) chain of arguments from givens to conclusions
3	Ability to notice helpful facts and make some inferences, but inability to be economic or precise, excess facts and/or imprecise labelling used leading to circuitous or clouded chains of argumentation
4	Ability to notice helpful facts, make inferences and coherent chain of arguments from givens to conclusions efficiently/economically

5. Results and discussion of results

5.1 Quantitative results

The fact that of the 170 educators 94 were females and 76 males (55% and 45%, respectively) was a positive indication of a higher rate of participation from women teachers. However, when tested for significance, the *Chi*-square value of 1.906 for 1 degree of freedom led to the conclusion that the difference in participation rates was not statistically different at $p < 0.05$. Further quantitative analysis (*cf* Table 2) showed that the mean score for females was higher than that for males. However, this difference was not statistically significant. The overall standard deviations indicated greater dispersion among scores for females than those for males.

Table 2: Overall students' performance

Subjects	Mean Score	Standard deviation	Minimum	Maximum	Totals	%
Females	3.149	1.502	0	4	94	55
Males	3.053	1.487	0	4	76	45
Overall	3.124	1.468	0	4	170	100

Table 3 shows a further disaggregation of scores by gender and degree of success in solving the proof problem.

Table 3: Performance analysis by score and gender

Score	Females	%	Males	%	Total	%
4	66	70	49	64	115	68
3	8	9	9	12	17	10
2	3	3	3	4	6	4
1	2	2	6	8	8	5
0	15	16	9	12	24	14
Total	94	100	76	100	170	100

5.2 Qualitative results

Four categories of misconceptions were identified in the investigation. Geometric proofs are normally presented in a two-column format that has a statement and a reason, and this is a long-standing tradition in South African geometry education (Fish 1996). The layout of the students' proofs evidenced this tradition as the following examples show.

5.2.1 A misconception that 'listing of properties of geometric shapes = proof'

This misconception indicates lack of basic geometrical knowledge and vocabulary.

Type IA misconception: Student 1 (Margaret)'s response.

Given: (1) AD // EC

(2) ∠ BAD = ∠ DAC

RTP: AB//DE

Statement	Reason	Comment
L_1: ABD is Δ:	AB//ED in Δ EDC	(But AB is not in Δ EDC)
L_2: AD = ED	D is a mid-point in Δ ADE	(D is not a midpoint but a vertex of Δ ADE)
L_3: BD = DC	Adjacent angles	(BD and DC are not angles but chords substending equal angles at the circumference)
L_4: AB // ED	Opposite sides of two Δ s	(A Δ cannot have opposite sides. This is what has to be proved)
L_5: D is a midpoint in Δ ADC		(It's a vertex, not a mid-point) – repetition
L_6: ABDC is a trapezium		(No reason given, none of the givens imply that – simply making an assumption.)
L_7: ∴ AB // ED.		(NB: L_n refers to Line n of the proof)

- Analysis of Margaret's proof

Margaret's[3] solution evidenced serious lack of basic geometrical knowledge and vocabulary. She could not accurately identify the sides of a triangle (L_1), twice she mixed up the concept of mid-point

3 Names used in this article are fictive and not the real names of the students participating in this study.

with that of vertex (L_2 and L_5), and mixed-up the concept of side with that of angle (L_3). She also referred to opposite sides of two triangles as parallel (L_4), which is impossible, and further considered ABCD to be a trapezium (L_6) without justification when it was not. Unsurprisingly, the relevant notion of equality of alternate interior angles was not in evidence, so the given information about parallel lines and transversals was not made use of. Nor was the theorem about equal chords subtending equal angles at the circumference employed harnessed. The given facts that AD//EC, and $\angle$ BAD = $\angle$ BAC were not put to use either. In other words, there was no evidence of adequate knowledge of basic geometric objects, definitions, and relationships and let alone parallel line and circle geometry theorems. Although the student was aware that there ought to be a reason for each statement, she frequently offered false reasons. The ability to make inferences was therefore evidently absent. The student also used a logically invalid procedure by citing the theorem to be proved in her proof, an error which has also been identified in earlier studies and has been attributed to the lack of teaching emphasis on the meaning of proof (Gagatsis & Demetriadou 2001, Senk 1985). The so-called learning paradox suggests that concepts make sense only if we have some abstract schema (ideas, vocabulary, and symbolism) to organise and give meaning to the concept to be learnt (Wheatley 1995). The prerequisite concepts are such a schema or "search space" and Margaret's level of response was could be classified as van Hiele Level 1 (van Hiele 1986), signifying a very weak "initial search space" or "set of relevant facts known by the individual" (Schoenfeld 1985: 14). She could accordingly not score any marks.

- Pedagogical implications of Margaret's proof

The seemingly extreme lack of basic knowledge of geometry calls attention to the need to return to basics in order to help students like Margaret. In their study, Gagatsis & Demetriadou (2001) also found that a high percentage of errors committed by learners in geometry proof problems concerned knowledge or handling of theory because classical geometry demands a good knowledge of a large part of the theory (definitions, theorems). In their opinion, "to know substantially" implicitly meant "to memorise" and consequently "to remember", but not necessarily to construct in one's own way. This initial step is frequently condoned as a starting point to build students' prior knowledge with basic facts which they can recall spontaneously when needed. The memorisation process would need to be preceded by or promptly followed up with an emphasis on understanding to prevent negative influence on students' achievement in classical Euclidean geometry. South Africa is a case in point where learners were traditionally made to memorise "rider after rider" with little understanding (Fish 1996) and Margaret is a likely product of the era of traditional geometry education with a great deal of complicated deductive theory that was meaninglessly memorised. She relied on what Simon (1996: 200) refers to as "bits of recalled knowledge". From the revealed knowledge gaps we add that her recalled bits were themselves misunderstood knowledge of triangles which could not help her to see the geometry of the circle, parallel lines and let alone a combination of both.

Although the student appeared to be completely out of depth, a silver lining was that she did something. Wheatley (1995: 4) contends that the most fundamental problem-solving heuristic is for a student to do something. Although, as already noted, she was frequently aware of the need for statements to be justified, she was still at a very elementary level of understanding vocabulary and properties of shapes. This suggested remedying similarly affected students by familiarising them with basic concepts/properties/definitions such as angles, vertices, and sides of a triangle, chord/circumference of a circle, angle subtended at the circumference and parallelism accompanied by appropriate visualisation and labelling/naming conventions.

5.2.2 A misconception that 'any property or relationships assumed true = proof'

This misconception indicates that use of inappropriate frame of reference (or geometrical knowledge).

Type IB misconception: Student 2 (Sibongile)'s response.

Given: (1) AD // EC

(2) $\angle$ BAD = $\angle$ DAC

RTP: AB//DE

Statement	Reason	Comment
In Δ ABD and Δ DEA		
L_1: AD = AD	Common side	(True but unhelpful fact)
L_2: ∠ B = ∠ E	Opp ∠ s of a cyclic	(Supplementary ≠ equality) quad are supplementary
L_3: ∠ D = ∠ A	AD//EC	Not clear which ∠ D or which ∠ A)
L_4: ∴ Δ ABD /// Δ DEA.		(So ∠ BAD = ∠ ADE?)
L_5: ∴ AB // ED		(So similarity ⇒ corresponding sides are parallel?)

- Analysis of Sibongile's proof

Sibongile correctly deduced that, if a side is common to two shapes, then it is the same length (L_1). This was true but redundant in the circumstances of the question. She was also correctly aware that opposite angles of a cyclic quadrilateral are supplementary (L_2), but used this fact to justify equality of opposite angles which was an incorrect deduction/inference. The naming of angles was done ambiguously (L_2 and L_2). There were five ∠ A's (∠ BAD, ∠ BAC, ∠ BAE, ∠ DAC, and ∠ CAE). Similarly, there were five ∠ D's which did not make the naming of angles helpful. Although the student was correct to conclude that Δ ABD was similar to Δ DEA (L_4) by the angle-side-angle (ASA) postulate for congruency/similarity, this was incorrectly applied to justify that AB was parallel to ED (L_5) (In itself this signified partly remembered knowledge of the fundamental theorem of similarity.) The "givens" were not used as possible points of departure. In the process, the student did not notice the very foundation upon which a proof could have been built. The student did not seem to consider the rationale for the givens in the first place and thus used an inappropriate frame of reference (congruency and similarity). Kim & Hannafin (2010) point out that student difficulties emanating from limited prior knowledge and experience can lead to cognitive overload. We presume the use of an inappropriate frame of reference and failure to observe basic geometrical object labelling conventions to be a part of such prior knowledge. Sibongile's level of response was also classified as Van Hiele Level 1 (Van Hiele 1986), signifying an incorrect 'initial search space' and she could not score any mark.

- Pedagogical implications of Sibongile's proof

Sibongile appeared to be a more redeemable case than Margaret in that some of her statements were true. A remedial programme that starts with exploring the consequences for givens to identify and select an appropriate frame of reference could be appropriate in scaffolding such students. Opportunities could be created for students to brainstorm the consequences of each of the givens in small groups in order to generate a collection of choices. For example, if AD// EC questions to generate a repertoire of choices could be: Which straight lines (transversals) intersect/meet with both parallel lines? What facts (theorems/riders/postulates) do we know or can we deduce about angles formed at the intersection of the parallel lines and the transversal(s)? Essentially, this would entail refreshing the learners' knowledge stock of geometrical objects, properties and relationships between objects/properties (Van Hiele level 2) to scaffold them.

Since students view the prospect of proving through the lens of their existing stock of geometrical knowledge, teachers' attempts at communication about deductive proof should not necessarily or always result in conveying a ready-made meaning. Instead, such communication efforts should evoke sense-making and allow for different meanings to emerge. If the key role of proof is the promotion of mathematical understanding, as Hanna (2000: 5) suggests, then it would be wise to heed the constructivists and help students like Sibongile to develop a repertoire of relevant geometric vocabulary, concepts, definitions and theorems which they can conveniently evoke later to hypothesise and conjecture during their deductive geometric proving activities. In the context of the proof problem being analysed, it would mean concepts such as parallel lines, transversal, alternate angles (to be compared and contrasted with corresponding, complementary, and vertically opposite angles), alternate angle theorem, circle circumference, chord, arc, angle subtended at the circumference, angles subtended by equal arcs theorem, and so on. Constructivists believe that students see what they understand (Wheatley 1991). This simply suggests that students cannot see alternate angles or angles subtended by the same arc if they do not understand what they are.

5.2.3 A misconception that 'restating or supplying own givens = proof'

This misconception suggests on ability to recognise some helpful facts but inability to make a logical deduction.

Type II Misconception: Student 3 (Rethabile)'s response.

Rethabile presented the following as her proof:

Given: (1) AD // EC

(2) ∠ BAD = ∠ DAC

RTP: AB//DE

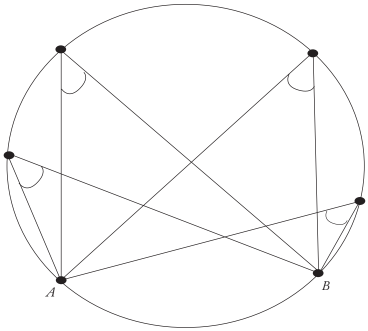
Statement	Reason	Comment
L_1: ∴ AD // EC	Given	(A given stated as a conclusion, odd to start with ∴)
L_2: ∠ BAD = ∠ DAC	Given	(Restating a given)
L_3: ∠ A = ∠ E	(∠ s subtended by same chord)	(True only if ∠ A and ∠ E are those subtended by chord DC, symbolisation problem)
L_4: ∴ ∠ BAD = ∠ DEC		(True if ∠ BAD = ∠ DAC = ∠ DEC is transitively subsumed, communication problem)

Statement	Reason	Comment
L_5: ∠ B = ∠ C	Right ∠ d Δ	(Unclear which ∠ B, ∠ C or right ∠ d Δ. Could be true for ∠ DBA and ∠ ACD only if AD was given to be a diameter)
L_6: ∴ AB//ED		(reason above inaccurate)

- Analysis of Rethabile's proof

Rethabile apparently did not use any of the givens. She simply restated them as conclusions (L_1 and L_2) and thus could not advance her argument/proof in a meaningful way. The restatement signified an awareness of the givens but that awareness was apparently not harnessed any further. She was correctly aware (L_3) of the theorem which states that "... for any chord ... in a circle all ... the angles subtended by points anywhere on the same semi-arc of the circle will be equal" (Artmann 2009) (*cf* Figure 1). This was one of two theorems which were helpful to the development of the proof. However, she did not name or designate her angles precisely enough and it was difficult to know which ∠ A or ∠ E she referred to. The reader was left to fill the gaps by assuming the angles to be those subtended by chord DC (∠ CAD was equal to ∠ CED). She then concluded that ∠ BAD was equal to ∠ DEC (L_4) which muddled the meaning of ∠ A and ∠ E. She was awarded one mark for her efforts this far.

Developing her proof further, Rethabile claimed (L_5) that ∠ B was equal to ∠ C (both insufficiently referenced again) because they were in right-angled triangles. Both the deduction and premise were incorrect. Using these incorrect facts she inferred that line segments AB and ED were parallel (L_6). In other words, she did not demonstrate adequate additional background knowledge (for instance, relationship between parallel lines, transversals, and alternate angles) with which to make further progress. The alternate angle theorem was thus used to show neither that ∠ DAC was equal to ∠ ACE nor, in combination with Thales's theorem, that ∠ ADE and ∠ BAD were both alternate and equal. Earlier studies of how students prove have also stressed the importance of maintaining the connections between proving and knowing (Herbst 2002a). With these shortcomings Rethabile was unable to score any further marks. Her response could be classified as Van Hiele Level 1.



Thales of Miletus is generally credited with giving the first proof that for any chord *AB* in a circle, all of the angles subtended by points anywhere on the same semi arc of the circle will be equal.

Figure 1: Thales of Miletus's Theorem (fl c 600 BC)

- Pedagogical implications of Rethabile's proof

Rethabile's insufficient or imprecise designation of angles implied a weakness in communication skills as an obstacle to effective handling of deductive geometric proof. Learners such as Rethabile who seem to have some correct ideas in some instances, but cannot express themselves adequately/accurately on paper need to be assisted to gain precision in their references to geometric objects by encouraging them to critically reflect on the meanings of the symbols they use and to search for alternative or unintended interpretations that may arise from them. Rethabile's failure to justify statements with correct reasons (for example, right-angled triangle) together with the failure to make a proper link between one step of a proof and the subsequent step implied a lack of effective argumentation skills for proof execution. Learners encountering such difficulties need to be encouraged to read their (sequence of) statements again and again and to critically "listen" to the coherence in their argument – a critical metacognitive skill.

Rethabile's failure to use Thales Theorem effectively in other parts of her proof when she had already proven to be aware of it was a sign of inadequate resource management skills. Students encountering such difficulties need to be encouraged to view a proof problem from different angles throughout the activity and to constantly experiment with the length and breadth of the consequences of the givens. Even as he indicts traditional instruction and testing for providing little opportunity for students to demonstrate the breadth and depth of their misconceptions or weakness in resources, Schoenfeld (1985: 12) suggests more explicitly that "... to be efficient, students need coaching in how to manage resources at their disposal".

5.2.4 A misconception that 'making correct inferences in any order = proof'

This misconception indicates an ability to notice some helpful facts and make an inference, but inability to organise information in a logical chain of arguments leading from givens to conclusions.

Type III misconception: Student 4 (Sandile)'s proof.

Given: (1) AD // EC

(2) $\angle$ BAD = $\angle$ DAC

RTP: AB//DE

Statement	Reason	Comment
L_1: BD = DC	Subtended by equal $\angle$ s	(Sees the wood but cannot see the forest)
L_2: $\Rightarrow \angle$ DAC = $\angle$ ACE	Alternate $\angle$ s	(Correct)
L_3: $\angle$ BAD = $\angle$ CED	Alternate $\angle$ s	(Correct, subtended by equal arcs but unhelpful)

Statement	Reason	Comment
L_4: ∠ ADE = ∠ BAD	Alternate ∠ s	(Not all alternate angles are equal, but showing this correctly by other means is critical to the required proof)
L_4: ∠ ADE = ∠ BAD	Alternate ∠ s	(Not all alternate angles are equal, but showing this correctly by other means is critical to the required proof)
L_5: ⇒ ∠ ADE = ∠ ACE		
L_6: ∴ AB // ED	Alternate ∠ s are equal.	(Incorrect if the angles are those in the previous step above)

- Analysis of Sandile's proof

Sandile appeared to be aware of some helpful facts (for example, BD = DC, ∠ DAC = ∠ ACE (L_1 and L_2)) and was able to make some valid (but not always necessarily helpful) inferences (L_3 and L_6). He encountered some difficulties in organising the deductions into logical chains of arguments leading from givens to conclusions. For example, he gave alternate angles as a reason for ∠ BAD = ∠ CED (L_3) which was inaccurate. He also made a statement and gave a reason the other way round when claiming that ∠ ADE = ∠ BAD because they were alternate (L_4), instead of showing that the angles are equal first and then alternate, a valid and necessary condition for line segments AB and ED to be parallel. This evidenced difficulty in correctly sequencing statements and reasons. He made no further use of the equality of angles subtended by the same arc (Thales's theorem), although he was aware of its corollary that chords subtending equal angles were equal (first statement). He was a redeemable case in that he had some of what Watts (1994: 13) refers to as an appropriate framework of prior knowledge. However, he lacked a firm grasp of the prior knowledge of alternate angles as shown by the conclusion that ∠ ADE = ∠ ACE on the premise that they were alternate when in fact they were not. The fact that the middle angle (∠ DAC) was both equal to an alternate angle (∠ BAD) and to another angle (∠ DEC) subtended by the same arc as itself formed the heart of the proof. In fact, this is an application of one of Euclid's five common notions stating that "... things equal to the same thing are equal" (Artmann 2009). For being aware of some helpful facts (for example, all pairs of angles identified as equal were indeed equal) and being able to draw some helpful inferences, Sandile was awarded two marks. His response could be classified as Van Hiele Level 2.

- Pedagogical implications of Sandile's proof

Pedagogically students like Sandile can be encouraged to be always on the lookout for or be able to construct sequences of geometric statements/premises (postulates, propositions, theorems, and so on) which make one thing simultaneously congruent to two (or more) other things but for different reasons. Such awareness bears the potential to significantly improve students' deductive proof handling competencies in geometry education as congruency plays a central role in many geometric proving tasks. Ernest (2007), for example, shows that proof and calculation can be logically equivalent where transitivity is involved in a sequence of proof statements such as $n_1=n_2$, $n_2=n_3$, $n_3=n_4$ leading to the conclusion that $n_1=n_4$. Students might be well equipped but lack the ability to use the right tools at the right time, and in the correct sequence, to obtain the desired proof outcome. As in the case of the first part of the previous student's proof, this was a resource management problem or structuring weakness, implying that students should be encouraged and guided to see one step of a proof as a logical outcome or consequence of the preceding steps, then as a justification for the next resulting in a coherent or carefully connected argument. In such instances, the deductive style of teaching proof to students directly might be advisable if the "... reason for the learning of proof is to learn to question the truth of statements" (Hemmi 2010: 283).

5.2.5 A misconception that 'long proof = good proof'

This misconception indicates an ability to notice helpful facts and make some inferences, but inability to be efficient-excess, repeated facts or long route noticed leading to circuitous chains of arguments.

Type IV misconception: Student 5 (Jacques)'s proof.

Given: (1) AD // EC

(2) ∠ BAD = ∠ DAC

RTP: AB//DE

Statement	Reason	Comment
L_1: Let ∠ BAD = ∠ DAC = x		(Algebraic skills applied = using a hammer to crack a nut?)
L_2: ∠ DAC = ∠ ACE	Alt ∠ s, AD//EC	(Correct and necessary)
L_3: ∠ BAD = ∠ BCD	Chord BD	(Correct but not necessary)
L_4: ∠ ECA = ∠ EDA	Chord AE	(Correct but not necessary)
L_5: ∠ ADE = ∠ DEC	Alt ∠ s are equal: AD//EC	(Correct but not absolutely necessary)
L_6: ∴ ∠ EOA = $2x$= (∠ OEC + ∠ OCE), Ext ∠ s of Δ OEC		
L_7 and		
L_8: ∠ BAC = ∠ AOE = $2x$		(True only if 'O' assumed as above)
L_9: So if ∠ BAC = ∠ AOE, then BA//ED, (Alt ∠ s)		(True if 'O' assumed to be the intersection of AC and DE)

- Analysis of Jacques's misconception and implications

Jacques could notice some helpful facts and make correct logical deductions. For example, alternate angle theorem used to justify equality of ∠ DAC and ∠ACE (L_2). She however, presented additional information, which contained correct statements and inferences but was not helpful. For example, although Thales's theorem was needed in the solution, it was applied to show that ∠ BAD was equal to ∠ BCD and ∠ ECA equal to ∠ EDA which were wasteful applications (L_3 and L_4). This wastefulness served to lengthen and obfuscate the proving process. In the flurry, she ironically did not have the presence of mind to apply the theorem to show that ∠ DAC was equal to ∠ DEC which was necessary to quickly arrive at the conclusion in L_6. The conclusion in L_6 was itself extraneous information (the sum of any two interior angles of a triangle is equal to the exterior opposite angle) (L_6) seemed to work effectively if the reader guessed point O correctly. That point O referred to in the proof was not labelled and the reader was left to guess if it was the centre of the circle or simply a point of intersection led to ambiguity in associated argument lines (L_6, L_8 and L_9). Watts (1994) affirms that sometimes too much information concerning the problem can be a hindrance to its efficient solution. Although excess information tended to muddle the proof, the student could be commended for a different kind of solution (algebraic), uneconomic though it might have been. Her response was awarded three marks and was deemed to be at Van Hiele Level 3 as she could perceive relationships between properties (alternate angle theorem, Thales's theorem, and the exterior angle theorem of a triangle) but did not display adequate command of necessary and sufficient conditions characteristic of Level 4.

- Pedagogical implications of Jacques's proof

Jacques's proof underscored the importance of designing or selecting open-ended proof problems amenable to solution by multiple methods. Such problems provide learners with the opportunity to compare and contrast different solution strategies and evaluate them for effectiveness and efficiency, an important metacognitive skill in deductive geometric proof. Educators should be encouraged to select proof problems that lend themselves to multiple solution strategies not only for the pedagogical value of comparing and selecting the most efficient strategies, but also to emphasise that textbook mathematics is man-made and therefore a fallible human activity to be approached with an open, critical mind. In other words, although economy (recognition of necessary and sufficient conditions) is essential for the elegance of proof, alternative proofs are to be accepted and encouraged as subtle methods of deepening learning in students.

In his argument for a delay of structure in learning and problem-solving situations to allow for productive failure, Kapur (2009) points to the efficacy of learner-generated structures – comprehensions, conceptions, representations, and understandings – even though these may not be correct initially and the processes of arriving at them not as efficient. To eventually achieve economy learners can be encouraged to critically evaluate each step of their argumentation according to the criteria of 'necessary and sufficient conditions'.

5.3 Implications for teacher professional development

Lloyd (2002: 149) cautions that perhaps the greatest obstacle for teachers is a lack of personal familiarity with mathematical problem-solving and sense-making – processes which the majority of them have never experienced as students. The same caution should certainly apply to deductive geometric proof which has often been used to scare away learners from mathematics. In a recent study, Ndlovu (2011: 1407) notes that some teachers question why geometry is being re-introduced in the mathematics curriculum since it "was always a problem" for them to teach previously, and consequently they plead for support. Enhanced teachers' knowledge of students' misconceptions should go a long way in equipping them for effective delivery and guidance of proof activities in their classroom. The different levels of learner misunderstandings discussed in this article suggest that teachers can gain from being aware of the possible causes at each level and pre-emptively plan to identify the levels appropriately and to consequently implement appropriate instructional measures. For example, when it is clear that a student is in the Type I category, it would be futile to start emphasising the sequencing of statements to make coherent chains of arguments as the learners will be lacking the appropriate basic concepts frame of reference.

When students can see some of the helpful facts but cannot make inferences or deductions (Type II category), it shows that they are ready for the teacher's intervention at the level of theory where the consequences of givens are explored experimentally at first to generate a repertoire of choices. When learners can suggest a range of inferences (Type III category) it then makes sense for the teacher to intervene by encouraging a critical evaluation of consequences for usefulness.

When learners can make useful (and wasteful) inferences from an appropriate frame of reference (Type IV category) they can then be helped to link chains of statements to the conclusions by exploiting corollaries (necessary conditions) that make the conclusion true in an economic way. When learners can make several proving strategies, they can then be engaged in the critical evaluation of each of them for efficiency. This implies that the geometry classroom should be characterised not merely by practical demonstrations and visualisations but also by an appropriate sequencing of problems according to levels of complexity, ranging from one-step to hypothetical or multi-step problems identified by Heinze *et al* (2008), as well as by the adoption of a discursive, dialectical classroom discourse, where learners are given opportunities to conjecture, hypothesise, test and prove or refute their own conjectures and hypotheses, and those of others. Such approaches resonate with the different roles and justifications for the teaching of proof alluded to at the beginning of this article, namely formal, utilitarian, mathematical and intuitive modal arguments (Gonzalez & Herbst 2006); explanation, verification, discovery, intellectual challenge and systematisation roles (De Villiers 1999); training in logical and critical thinking (Deckerson & Doer 2008). To aid the discursive classroom discourses teachers could be exposed to dynamic geometry software such as *Sketchpad*, *Cabri* and *Geogebra* during their initial as well as in-service training to develop ICT competencies in real-time visualisation, demonstration, experiment and conjecturing with geometrical objects, relationships, definitions, and theorems to make the learning of deductive geometrical proof more interesting and accessible to a greater number of learners.

6. Conclusion

This article presented the many reasons for teaching proof, in general, and deductive proof, in particular, as being to enable students to develop thinking skills that are important within mathematics and transferable to other areas of inquiry. Because of this perceived value, a high pedagogical premium was placed on students' mistakes or misconceptions. To this end we selected and analysed examples of what we termed misconceptions or alternative conceptions about deductive geometric proof. A tentative taxonomy, subject to further refinement, was proposed and possible remedial strategies proffered. This study also illustrated that, although proof is a "tired" research topic, we can still derive some new insights from the nature of errors students make, if only to emphasise that the image of "perfect mathematical knowledge" often presented to students in the traditional classroom is a façade that denies them the opportunity to re-invent mathematics. The initial comments we made in the last student's proof suggest that there may be many, out there, who are still locked up in the traditional one-answer-one-method model of teaching mathematics.

Memorising a proof without understanding the interconnectedness (the logical relationship) between one statement and the next can only serve to stifle understanding and to alienate students not merely from deductive geometric proof but also from mathematics in general. In this ongoing study we can conclude for now that formative assessment, like homework, can be used not only to locate mistakes, but also to figure out why they were made, how teachers might deal with them, and how they could provide support to students by way of further explanation and tutoring (Fang 2010). This approach is supported by the observation made by Pedrosa de Jesus *et al* (2005: 182) that we can learn some pedagogical lessons from exploring the content of students' procedural knowledge and understanding. In this case it is the procedural knowledge and understanding of deductive geometric theorem proving *vis-à-vis* a well-indexed knowledge base of theory and procedural etiquette in proving. That is, when students make mistakes, they must be considered opportunities for reconstruction of their knowledge or an inventory of all the facts, procedures, and skills, which the individual is capable of bringing to bear on a particular proof problem, rather than the consequence of ignorance (Hobden 1998, Schoenfeld 1985).

This article has limitations. First, since a single item was used, the results need to be interpreted with that limitation in mind. Secondly, since this was an achievement test, students could not be interviewed to shed more light on the thinking behind their decisions and statements. Nevertheless, the multi-step character and the duality of the contributory concepts required (parallelism and circle theorems in plane geometry) served to typify deductive geometric proofs in high school geometry.

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